Canonical singular hermitian metrics on relative log canonical bundles

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Abstract

We introduce a new class of canonical analytic Zariski decompositions (AZD's in short) called the supercanonical AZD's on the canonical bundles of smooth projective varieties with pseudoeffective canonical classes. We study the variation of the supercanonical AZD $\hat{h}_{can}$ under projective deformations and give a new proof of the invariance of plurigenera. Moreover extending the results to the case of KLT pairs, we prove the invariance of logarithmic plurigenera for a family of KLT pairs. This paper supersedes [T3, T6].
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Contents

1 Introduction

Let X be a smooth projective variety and let K_X be the canonical bundle of X. In algebraic geometry, the canonical ring $R(X, K_X) := \oplus_{m=0}^{\infty}\Gamma\left(X, \mathcal{O}_X(mK_X)\right)$ is one of the main objects to study. And it has been studied the variation of pluricanonical systems in terms of variation of Hodge structures([F, Ka1, V1, V2]).

The purpose of this article is to study the variation of (log) canonical rings on a projective family by introducing a canonical singular hermitian metric on the relative (log) canonical bundles. The important feature here is the semipositivity of the relative (log) canonical bundles and the invariance of (log) plurigenera is obtained at the same time. Moreover we can deal with the adjoint line bundle of a pseudoeffective $\mathbb{Q}$-line bundle in a systematic way.

Let X be a smooth projective variety such that K_X is pseudoeffective. In this article, we construct a singular hermitian metric $\hat{h}_{can}$ on K_X such that

(1) $\hat{h}_{can}$ is uniquely determined by X,

(2) The curvature current $\sqrt{-1}\,\Theta_{\hat{h}_{can}}$ is semipositive,

(3) $H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(\hat{h}^m_{can})) \simeq H^0(X, \mathcal{O}_X(mK_X))$ holds for every $m \geqq 0$,

where $\mathcal{I}(\hat{h}^m_{can})$ denotes the multiplier ideal sheaf of $\hat{h}^m_{can}$ as is defined in [N]. We may summerize the 2nd and the 3rd conditions by introducing the following notion.

Definition 1.1 (AZD)*([T6, T2]) Let M be a compact complex manifold and let L be a holomorphic line bundle on M. A singular hermitian metric h on L is said to be an analytic Zariski decomposition (AZD in short), if the followings hold.*

(1) The curvature current $\sqrt{-1}\,\Theta_h$ is semipositive.

(2) For every $m \geq 0$, the natural inclusion

$$H^0(M, \mathcal{O}_M(mL) \otimes \mathcal{I}(h^m)) \to H^0(M, \mathcal{O}_M(mL))$$

is an isomorphim. $\square$

Remark 1.2 *A line bundle L on a projective manifold X admits an AZD, if and only if L is pseudoeffective ([D-P-S, Theorem 1.5]).* $\square$

In this sense, we construct an AZD $\hat{h}_{can}$ on K_X depending only on X, when K_X is pseudoeffective (by Remark 1.2 this is the minimal requirement for the existence of an AZD). In fact $\hat{h}_{can}$ is not only an AZD of K_X, but also a singular hermitian metric with minimal singularities on K_X (cf. Definition 5.2). The important feature of this canonical metric $\hat{h}_{can}$ is that it naturally defines a singular hermitian metric on the relative canonical bundle on a smooth projective family of smooth projective varieties with pseudoeffective canonical bundles just by assigning the canonical metric on each smooth fiber and taking lower-semicontinuous envelope and extension across singular fibers (cf. Theorem 1.13). And the most

important fact is that the resulting canonical metric $\hat{h}_{can}$ has semipositive curvature on the total space of the family. This immediately gives a new proof of the invariance of plurigenera for smooth projective families (cf. Corollary 1.14). And this result implies the existence of a canonical hermitian metrics on the direct image of a relative pluricanonical system (cf. Theorem 4.15) with "Griffith semipositive" curvature. This semipositivity result is similar to [Ka1, V1, V2].

On the other hand, it is natural to consider not only a single algebraic variety but also a pair of a variety and a divisor on it. One of the important class of such pairs is the class of KLT pairs (cf. Definition 4.1). In general it is a basic philosophy that the most of the results for the absolute case (the case of smooth projective varieties) can be generalized to the case of KLT pairs. Such generalization is important because the log category is more natural to work. For example the induction in dimension sometimes works more naturally in the log category (see [B-C-H-M] for example).

In this paper we define a similar canonical singular hermitian metric on the log canonical bundle of a KLT pair with pseudoeffective log canonical divisor. And it satisfies a similar properties for a projective deformation of KLT pairs (cf. Theorem 4.3). By using this metric, we can deduce the invariance of logarithmic plurigenera (cf. Theorems 1.15 and 1.19) and also local freeness and semipositivity of the direct images of pluri log canonical systems (cf. Theorems 1.15 and 4.15).

Moreover the construction here is applicable to the case of general noncompact complex manifolds such as bounded domains in $\mathbb{C}^n$(cf. Section 2.6). This seems to be an interesting topic in future.

1.1 Canonical AZD h_{can}

If we assume that X has nonnegative Kodaira dimension, we have already konwn how to construct a canonical AZD for K_X. Let us review the construction in [T5].

Theorem 1.3 *([T5]) Let X be a smooth projective variety with nonnegative Kodaira dimension. We set for every point $x \in X$*

$$K_m(x) := \sup\left\{ \mid \sigma \mid^{\frac{2}{m}}(x)\,;\, \sigma \in \Gamma(X, \mathcal{O}_X(mK_X)), \left|\int_X (\sigma \wedge \bar{\sigma})^{\frac{1}{m}}\right| = 1\right\} \tag{1.1}$$

and

$$K_\infty(x) := \limsup_{m\to\infty} K_m(x). \tag{1.2}$$

Then

$$h_{can} := \text{the lower envelope of } K_\infty^{-1} \tag{1.3}$$

is an AZD on K_X. □

Remark 1.4 *By the ring structure of $R(X, K_X)$, we see that $\{K_{m!}\}$ is monotone increasing, hence*

$$\limsup_{m\to\infty} K_m(x) = \sup_{m\geqq 1} K_m(x)$$

holds. □

Remark 1.5 *Since h_{can} depends only on X, the number*

$$\int_X h_{can}^{-1}$$

is an invariant of X. But I do not know the properties of this number. □

Apparently this construction is very canonical, i.e., h_{can} depends only on the complex structure of X. We call h_{can} the **canonical AZD** of K_X. But this construction works only if we know that the Kodaira dimension of X is nonnegative apriori. This is the main defect of h_{can}. For example, h_{can} is useless to solve the abundance conjecture or to deduce the deformation invariance of plurigenera.

Moreover although h_{can} is an AZD of K_X, it is not clear that h_{can} has minimal singularities in the sense of Definition 5.2 below. But it is easy to see that h_{can} has minimal singularities, if K_X is abundant.

1.2 Supercanonical AZD $\hat{h}_{can}$

To avoid the defect of h_{can}, we introduce the new AZD $\hat{h}_{can}$. Let us use the following terminology.

Definition 1.6 (Pseudoeffectivity) *Let (L, h_L) be a singular hermitian $\mathbb{Q}$-line bundle on a complex manifold X. (L, h_L) is said to be pseudoeffective, if the curvature current $\sqrt{-1}\,\Theta_{h_L}$ of h_L is semipositive. And a $\mathbb{Q}$-line bundle L on a complex manifold X is said to be pseudoeffecive, if there exists a singular hermitian metric h_L on L with semipositive curvature.* $\square$

Let X be a smooth projective n-fold such that the canonical bundle K_X is pseudoeffective. Let A be a sufficiently ample line bundle such that for every pseudoeffective singular hermitian line bundle (L, h_L) on X, $\mathcal{O}_X(A+L)\otimes\mathcal{I}(h_L)$ and $\mathcal{O}_X(K_X+A+L)\otimes\mathcal{I}(h_L)$ are globally generated. The existence of such an ample line bundle A follows from Nadel's vanishing theorem ([N, p.561]). See Propositon 5.1 in Section 5.1 for detail.

For every $x \in X$ we set

$$\hat{K}^A_m(x) := \sup\left\{ \mid \sigma \mid^{\frac{2}{m}} (x) \mid \sigma \in \Gamma(X, \mathcal{O}_X(A + mK_X)), \parallel \sigma \parallel_{\frac{1}{m}} = 1 \right\}, \tag{1.4}$$

where

$$\parallel \sigma \parallel_{\frac{1}{m}} := \left| \int_X h_A^{\frac{1}{m}} \cdot (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right|^{\frac{m}{2}}. \tag{1.5}$$

Here $\mid \sigma \mid^{\frac{2}{m}}$ is not a function on X, but the supremum is takan as a section of the real line bundle $\mid A\mid^{\frac{2}{m}} \otimes \mid K_X\mid^2$ in the obvious manner[1]. Then $h_A^{\frac{1}{m}} \cdot \hat{K}^A_m$ is a continuous semipositive (n,n)-form on X. Under the above notations, we have the following theorem.

Theorem 1.7 *We set*

$$\hat{K}^A_\infty := \limsup_{m\to\infty} h_A^{\frac{1}{m}} \cdot \hat{K}^A_m \tag{1.6}$$

and

$$\hat{h}_{can,A} := \text{the lower envelope of } (\hat{K}^A_\infty)^{-1}. \tag{1.7}$$

Then $\hat{h}_{can,A}$ is an AZD of K_X. And we define

$$\hat{h}_{can} := \text{the lower envelope of } \inf_A \hat{h}_{can,A}, \tag{1.8}$$

where inf *denotes the pointwise infimum and A runs all the ample line bundles on X. Then $\hat{h}_{can}$ is a well defined AZD on K_X with minimal singularities (cf. Definition 5.2) depending only on X.* $\square$

Remark 1.8 *I believe that $\hat{h}_{can,A}$ is already independent of the sufficiently ample line bundle A.* $\square$

Remark 1.9 *In [T4], I have defined a similar AZD of K_X for a compact Kähler manifold X with pseudoeffective canonical bundle. The construction is even simpler than $\hat{h}_{can}$. But I have not yet proven the semipositivity property corresponding to Theorem 1.12 below in the case of Kähler deformations.* $\square$

Definition 1.10 (Supercanonical AZD) *We call $\hat{h}_{can}$ in Theorem 1.7 the supercanonical AZD of K_X. And we call the semipositive (n,n)-form $\hat{h}^{-1}_{can}$ the supercanonical volume form on X.* $\square$

Remark 1.11 *Here "super" means that corresponding volume form $\hat{h}^{-1}_{can}$ satisfies the inequality :*

$$\hat{h}^{-1}_{can} \geqq h^{-1}_{can}, \tag{1.9}$$

if X has nonnegative Kodaria dimension (cf. Theorem 2.9). $\square$

In the statement of Theorem 1.7, one may think that $\hat{h}_{can,A}$ may depend of the choice of the metric h_A. But later we prove that $\hat{h}_{can,A}$ is independent of the choice of h_A (cf. Lemma 2.6).

[1] We have abused the notations $|A|$, $|K_X|$ here. These notations are similar to the notations of corresponding linear systems. But we shall use the notation if without fear of confusion.

1.3 Variation of the supercanonical AZD $\hat{h}_{can}$

Let $f : X \longrightarrow S$ be a fiber space such that X, S are complex manifolds and f is a proper surjective projective morphism with connected fibers. Suppose that for every regular fiber $X_s := f^{-1}(s)$, K_{X_s} is pseudoeffective [2]. In this case we may define a singular hermitian metric $\hat{h}_{can}$ on $K_{X/S}$ similarly as above. Then $\hat{h}_{can}$ have nice properties on $f : X \longrightarrow S$ as follows.

Theorem 1.12 *Let $f : X \longrightarrow S$ be a proper surjective projective morphism with connected fibers between complex manifolds such that for every regular fiber X_s, K_{X_s} is pseudoeffective. We set S° be the maximal nonempty Zariski open subset of S such that f is smooth over S° and $X^\circ = f^{-1}(S^\circ)$. Then there exists a unique singular hermitian metric $\hat{h}_{can}$ on $K_{X/S}$ such that*

(1) $\hat{h}_{can}$ has semipositive curvature on X,

(2) $\hat{h}_{can}|X_s$ is an AZD of K_{X_s} with minimal singularities for every $s \in S^\circ$,

(3) For every $s \in S^\circ$, $\hat{h}_{can}\,|\,X_s \leqq \hat{h}_{can,s}$ holds, where $\hat{h}_{can,s}$ denotes the supercanonical AZD of K_{X_s}. And $\hat{h}_{can}|X_s = \hat{h}_{can,s}$ holds outside of a set of measure 0 on X_s for almost every $s \in S^\circ$.

We call $\hat{h}_{can}$ in Theorem 1.13 the relative supercanonical AZD *on $K_{X/S}$.* $\square$

To prove Theorem 1.12, first we shall prove the following slightly weaker version.

Theorem 1.13 *Let $f : X \longrightarrow S$, S° and $X^\circ := f^{-1}(S^\circ)$ as in Theorem 1.12.*
Then there exists a unique singular hermitian metric $\hat{h}_{can}$ on $K_{X/S}$ such that

(1) $\hat{h}_{can}$ has semipositive curvature on X,

(2) $\hat{h}_{can}|X_s$ is an AZD on K_{X_s} for every $s \in S^\circ$,

(3) There exists the union F of at most countable union of proper subvarieties of S° such that for every $s \in S^\circ \setminus F$, $\hat{h}_{can}\,|\,X_s \leqq \hat{h}_{can,s}$ holds, where $\hat{h}_{can,s}$ denotes the supercanonical AZD on K_{X_s}. And $\hat{h}_{can}|X_s = \hat{h}_{can,s}$ holds outside of a set of measure 0 on X_s for almost every $s \in S^\circ$. $\square$

The only difference between Theorems 1.13 and 1.12 is the existence of the set F in Theorem 1.13. We prove Theorem 1.12 by using Theorem 1.13 and the invariance of the twisted plurigenera: Corollary 3.11 below (cf. Corollary 3.12).

In Theorem 1.13, the assertions (1) and (2) are very important in applications. By Theorem 1.13 (or Theorem 1.12) and the L^2-extension theorem ([O-T, p.200, Theorem]), we obtain the following corollary immediately (To make sure we give a proof in Section 3.5).

Corollary 1.14 *([S1, S2]) Let $f : X \longrightarrow S$ be a smooth projective family over a complex manifold S. Then for every positive integer m, the m-genus $P_m(X_s) := \dim H^0(X_s, \mathcal{O}_{X_s}(mK_{X_s}))$ is a locally constant function on S* $\square$

1.4 Invariance of logarithmic plurigenera

In Section 4, we shall generalize Theorems 1.7 and 1.12 to the case of a projective families of KLT pairs (cf. Definition 4.1). See Theorems 4.2 and 4.3 below. As a consequence we have the invariance of logarithmic plurigenera:

Theorem 1.15 *Let $f : X \longrightarrow S$ be a proper surjective projective morphism between complex manifolds with connected fibers. Let D be an effective $\mathbb{Q}$-divisor on X such that*

(a) D is $\mathbb{Q}$-linearly equivalent to a $\mathbb{Q}$-line bundle (= a fractional power of a genuine line bundle) B,

[2] This condition is equivalent to the one that for some regular fiber X_s, K_{X_s} is pseudoeffective. This is well known. For the proof, see Lemma 3.5 below and Remark 3.6.

(b) The set: $S^\circ := \{s \in S|\ f$ *is smooth over* s *and* (X_s, D_s) *is KLT* $\}$ $(D_s := D\,|\,X_s)$ *is nonempty.*

Then for every positive integer m *such that* mB *is Cartier, the logarithmic* m*-genus:*

$$P_m(X_s, B_s) := \dim H^0(X_s, \mathcal{O}_{X_s}(m(K_{X_s} + B_s)))$$

is locally constant on S°*, where* $B_s := B|X_s$*. In particular the logarithmic Kodaira dimension of* (X_s, D_s) *is locally constant over* S° *and for such* m*,* $f_*\mathcal{O}_X(m(K_{X/S} + B))$ *is locally free over* S°*.* □

Remark 1.16 *It would be interesting to consider a flat family* $f : X \to S$ *such that the total space* X *has only canonical singularities and* $K_{X/S} + D$ *is* $\mathbb{Q}$*-Cartier and pseudoeffective. I believe that the present proof of Theorem 1.15 works also in this case.* □

We note that in the special case that B is a genuine line bundle, Theorem 1.15 has already been known ([C, Va]). In Theorem 1.15, the canonical choice of B is the minimal positive multiple of D so that the multiple has integral coefficients. But in general, some smaller positive multiple of D is $\mathbb{Q}$-linearly equivalent to a Cartier divisor. The following corollary is obvious.

Corollary 1.17 *Let* $f : X \longrightarrow S$*,*D *and* S° *be as in Theorem 1.15. Then we have that for every positive integer* m*,*

$$\dim H^0(X_s, \mathcal{O}_X(\lfloor m(K_{X/S} + D)\rfloor |X_s))$$

is locally constant on S° *and* $f_*\mathcal{O}_X(\lfloor m(K_{X/S} + D)\rfloor)$ *is locally free over* S°*.* □

1.5 KLT line bundles and invariance of plurigenera for adjoint line bundles

In the proof of Theorem 1.15, for $s \in S^\circ$, we consider the singular hermitian metric:

$$h_{D,s} := \frac{1}{|\sigma_D|^2}\,|X_s$$

on B_s (see (1.14) for the notation), where σ_D is a multivalued holomorphic section of B (see the convention below) with divisor D. The singular hermitian $\mathbb{Q}$-line bundle $(B_s, h_{D,s})$ is an example of the following notion.

Definition 1.18 *Let* (L, h_L) *be a singular hermitian* $\mathbb{Q}$*-line bundle on a smooth projective variety* X*.* (L, h_L) *said to be KLT (Kawamata log terminal), if the curvature current* $\sqrt{-1}\,\Theta_{h_L}$ *is semipositive and* $\mathcal{I}(h_L) = \mathcal{O}_X$*. For an open subset* U *of* X*, a pseudoeffective line bundle* (L, h_L) *on* X *is said to be KLT over* U*, if* $\mathcal{I}(h_L)|U = \mathcal{O}_U$ *holds.*

A $\mathbb{Q}$*-line bundle* L *on a smooth projective variety is said to be KLT, if there exists a singular hermitian metric* h_L *such that* (L, h_L) *is KLT.* □

Roughly speaking a KLT $\mathbb{Q}$-line bundle is a $\mathbb{Q}$-line bundle which admits a singular hermitian metric with semipositive curvature and relatively small singularities. In this sense, KLT $\mathbb{Q}$-line bundles are somewhere between semiample $\mathbb{Q}$-line bundles and pseudoeffective $\mathbb{Q}$-line bundles. The notion of KLT $\mathbb{Q}$-line bundles is a natural generalization of the notion of KLT pairs.

A very important example of KLT $\mathbb{Q}$-line bundle is the Hodge $\mathbb{Q}$-line bundle associated with an Iitaka fibtration. Let $f : X \to Y$ be an Iitaka fibration such that X, Y are smooth and f is a morphism. Then by [F-M, p.169,Proposition 2.2] $f_*\mathcal{O}_X(m!K_{X/Y})^{**}$ is invertible on Y for every sufficiently large m, where $**$ denotes the double dual. $f_*\mathcal{O}_X(m!K_{X/Y})^{**}$ is of rank 1 for every sufficiently large m. We define the $\mathbb{Q}$-line bundle

$$L := \frac{1}{m!}\,f_*\mathcal{O}_X(m!K_{X/Y})^{**} \tag{1.10}$$

on Y and call it the Hodge $\mathbb{Q}$-line bundle associated with $f : X \to Y$. And for every $y \in Y$ such that f is smooth over y, we set

$$h_L^{m!}(\sigma,\sigma)(y) = \left|\int_{X_y} (\sigma \wedge \overline{\sigma})^{\frac{1}{m!}}\right|^{m!} \qquad (\sigma \in L_y). \tag{1.11}$$

and call it the Hodge metric on L at y. Then h_L extends to a singular hermitian metric on L and (L, h_L) is KLT by the theory of variation of Hodge structures ([Sch]).

Using this new notion, we have a further generalization of Theorem 1.15 as follows.

Theorem 1.19 *Let $f : X \longrightarrow S$ be a proper surjective projective morphism between complex manifolds with connected fibers and let (L, h_L) be a pseudoeffective singular hermitian $\mathbb{Q}$-line bundle (cf. Definition 1.6 below) on X such that for a general fiber X_s, $(L, h_L)|X_s$ is KLT, We set*

$$S^\circ := \{s \in S|\, f \textit{ is smooth over } s \textit{ and } (L, h_L)|X_s \textit{ is well defined and KLT}\}.$$

Then for every positive integer m such that mL is Cartier, the twisted m-genus:

$$P_m(X_s, L_s) := \dim H^0(X_s, \mathcal{O}_{X_s}(m(K_{X_s} + L_s)))$$

is locally constant on S°, where $L_s := L|X_s$. $\square$

In fact Theorem 1.15 follows from Theorem 1.19 by taking (L, h_L) to be $(B, 1/|\sigma_D|^2)$. The main feature of Theorem 1.19 is that the singularities of h_L do not appear in the statement as long as the singularities are KLT. In this sense, KLT singularities are negligible in this case. The reason why we can neglect KLT singularities is that the construction of the AZD on $K_X + L$ (cf. Theorem 4.6) does not involve any high powers of h_L. In the special case that L is a genuine line bundle, Theorem 1.19 has already been known ([C, Va]). The main difficulty to deal with a singular hermitian $\mathbb{Q}$-line bundle is that if we take a multiple to make it a genuine line bundle, then we may get a nontrivial multiplier ideal sheaf.

1.6 A conjecture for Kähler fibrations

The invariance of plurigenera is an important consequence of Theorem 1.12 or Theorem 1.13. But anyway it has been already known by other methods. Actually one of the main significance of Theorem 1.12 is that it gives a perspective in the case of Kähler fibrations as follows.

Let $f : X \to S$ be a surjective proper Kähler morphism with connected fibers between connected complex manifolds. Let S° denote the complement of the discriminant locus of f. Let (L, h_L) be a hermitian line bundle on X with semipositive curvature. Suppose that $K_{X_s} + L|X_s$ is pseudoeffective for every $s \in S^\circ$. We shall consider an analogy of $\hat{h}_{can}$ as follows. For $s \in S^\circ$ we set

(1.12) $\qquad dV_{max}((L, h_L)|X_s) :=$ the upper semicontinuous envelope of

$$\sup\left\{h^{-1}|\ h\text{: a singular hermitian metric on } K_X \text{ such that } \sqrt{-1}\,(\Theta_h + \Theta_{h_L}) \geqq 0,\ \int_X h^{-1} = 1\right\},$$

where sup means the poitwise supremum. We call $dV_{max}((L, h_L)|X_s)$ the maximal volume form of X_s with respect to $(L, h_L)|X_s$. Then it is easy to see that $h_{min,s} := dV_{max}((L, h_L)|X_s)^{-1} \cdot h_L$ is an AZD of $K_{X_s} + L|X_s$ with minimal singularities (see Definition 5.2 and Section 5.2). This definition can be generalized to the case of noncompact complex manifolds such as bounded domains in $\mathbb{C}^n$. Now I would like to propose the following conjecture.

Conjecture 1.20 *In the above notations, we define the relative volume form $dV_{max,X/S}(L, h_L)$ on $f^{-1}(S^\circ)$ by $dV_{max,X/S}(L, h_L)|X_s := dV_{max}((L, h_L)|X_s)$ for $s \in S^\circ$ and we define the singular hermitian metric $h_{X/S}$ on $(K_{X/S} + L)|f^{-1}(S^\circ)$ by*

$$h_{min,X/S}(L, h_L) := \textit{the lower semicontinuous envelope of } dV_{max,X/S}(L, h_L)^{-1} \cdot h_L.$$

Then $h_{min,X/S}(L, h_L)$ extends to a singular hermitian metric on $K_{X/S} + L$ over X and has semipositive curvature. $\square$

We call $h_{min,X/S}$ the minimal singular hemitian metric on $K_{X/S} + L$ with respect to h_L. This conjecture is very similar to Theorem 1.12 and the recent result of Berndtsson([Ber]). If this conjecture is affirmative, we can prove the defomation ivariance of plurigenera for Kähler deformations. One can

consider also the case that (L, h_L) is pseudoeffective with KLT singularities. I have also other conjectures which relates the abundance of canonical bundle and the minimal singular hermitian metric on canonical bundle. I would like to discuss about Conjecture 1.20 and other conjectures in the subsequent papers.

The organization of this article is as follows. In Section 2, we prove Theorem 1.7. In Section 3, we prove Theorem 1.12 by using a result in [B-P, Corollary 4.2]. In Section 4, we generalize the reuslts in Sections 2 and 3 to the case of KLT pairs. Here the new ingredient is the use of dynamical systems of singular hermitian metrics.

Conventions

- In this paper all the varieties are defined over $\mathbb{C}$.
- We frequently use the classical result that the supremum of a family of plurisubharmonic functions locally uniformly bounded from above is again plurisubharmonic, if we take the upper-semicontinuous envelope of the supremum ([L, p.26, Theorem 5]).
- For simplicity, we denote the upper(resp. lower)semicontinuous envelope simply by the upper(resp. lower) envelope.
- In this paper all the singular hermitian metrics are supposed to be lower-semicontinuous.

Notations

- For a real number a, $\lceil a \rceil$ denotes the minimal integer greater than or equal to a and $\lfloor a \rfloor$ denotes the maximal integer smaller than or equal to a. We set $\{a\} := a - \lfloor a \rfloor$ and call it the fractional part of a.
- Let X be a projective variety and let D be a Weil divsor on X. Let $D = \sum d_i D_i$ be the irreducible decomposition. We set

(1.13) $$\lceil D \rceil := \sum \lceil d_i \rceil D_i,\ \lfloor D \rfloor := \sum \lfloor d_i \rfloor D_i,\ \{D\} := \sum \{d_i\} D_i.$$

- For a positive integer n, Δ^n denotes the unit open polydisk in $\mathbb{C}^n$ with radius 1, i.e.,

$$\Delta^n := \{(t_1, \cdots, t_n) \in \mathbb{C}^n; |t_i| < 1 (1 \leqq i \leqq n)\}.$$

  We denote Δ^1 simply by Δ.

- Let L be a $\mathbb{Q}$-line bundle on a compact complex manifold X, i.e., L is a formal fractional power of a genuine line bundle on X. A singular hermitian metric h on L is given by

$$h = e^{-\varphi} \cdot h_0,$$

  where h_0 is a C^∞-hermitian metric on L and $\varphi \in L^1_{loc}(X)$. We call φ the weight function of h with respect to h_0. We note that even though L is not a genuine line bundle, h makes sense, since a hermitian metric is a real object.

  The curvature current Θ_h of the singular hermitian $\mathbb{Q}$-line bundle (L, h) is defined by

$$\Theta_h := \Theta_{h_0} + \partial\bar{\partial}\varphi,$$

  where $\partial\bar{\partial}\varphi$ is taken in the sense of current. We define the multiplier ideal sheaf $\mathcal{I}(h)$ of (L, h) by

$$\mathcal{I}(h)(U) := \{f \in \mathcal{O}_X(U);\ |f|^2 e^{-\varphi} \in L^1_{loc}(U)\},$$

  where U runs open subsets of X.

- For a Cartier divisor D, we denote the corresponding line bundle by the same notation. Let D be an effective $\mathbb{Q}$-divisor on a smooth projective variety X. Let a be a positive integer such that $aD \in \mathrm{Div}(X)$. We identify D with a formal a-th root of the line bundle aD. We say that σ is a multivalued global holomorphic section of D with divisor D, if σ_D is the formal a-th root of a nontrivial global holomorphic section of aD with divisor aD. And $1/|\sigma_D|^2$ denotes the singular hermitian metric on D defined by

$$\frac{1}{|\sigma_D|^2} := \frac{h_0}{h_0(\sigma_D, \sigma_D)}, \tag{1.14}$$

where h_0 is an arbitrary C^∞-hermitian metric on D.

- For a singular hermitian line bundle (F, h_F) on a compact complex manifold X of dimension n. $K(K_X + F, h_F)$ denotes the diagonal part of the Bergman kernel of $H^0(X, \mathcal{O}_X(K_X + F) \otimes \mathcal{I}(h_F))$ with respect to the L^2-inner product:

$$(\sigma, \sigma') := (\sqrt{-1})^{n^2} \int_X h_F \cdot \sigma \wedge \overline{\sigma'}, \tag{1.15}$$

i.e.,

$$K(K_X + F, h_F) = \sum_{i=0}^{N} |\sigma_i|^2, \tag{1.16}$$

where $\{\sigma_0, \cdots, \sigma_N\}$ is a complete orthonormal basis of $H^0(X, \mathcal{O}_X(K_X + F) \otimes \mathcal{I}(h_F))$. It is clear that $K(K_X + F, h_F)$ is independent of the choice of the complete orthonormal basis.

Acknowledement: I would like to thank to the referee who suggested me the use of Fujita's elementary argument instead of Schmidt's theory of variation of Hodge structure([Sch]) in Section 3.1.

2 Proof of Theorem 1.7

In this section we shall prove Theorem 1.7. We shall use the same notations as in Section 1.2. The upper estimate of $\hat{K}^A_m$ is almost the same as in [T5], but the lower estimate of $\hat{K}^A_m$ requires the L^2-extension theorem ([O-T, O]).

2.1 Upper estimate of $\hat{K}^A_m$

Let X be as in Theorem 1.7 and let n denote $\dim X$. Let $x \in X$ be an arbitrary point. Let $(U, z_1, \cdots, z_n)$ be a coordinate neighborhood of X which is biholomorphic to the unit open polydisk Δ^n such that $z_1(x) = \cdots = z_n(x) = 0$.

Let $\sigma \in \Gamma(X, \mathcal{O}_X(A + mK_X))$. Taking U sufficiently small, we may assume that $(z_1, \cdots, z_n)$ is a holomorphic local coordinate on a neighborhood of the closure of U and there exists a local holomorphic frame $\mathbf{e}_A$ of A on a neighborhood of the closure of U. Then there exists a bounded holomorphic function f_U on U such that

$$\sigma = f_U \cdot \mathbf{e}_A \cdot (dz_1 \wedge \cdots \wedge dz_n)^m \tag{2.1}$$

holds. Suppose that

$$\left| \int_X h_A^{\frac{1}{m}} \cdot (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right| = 1 \tag{2.2}$$

holds. Then we see that

$$\begin{aligned} \int_U \mid f_U(z) \mid^{\frac{2}{m}} d\mu(z) &\leqq \left(\inf_U h_A(\mathbf{e}_A, \mathbf{e}_A)\right)^{-\frac{1}{m}} \cdot \int_U h_A\,(\mathbf{e}_A\,\mathbf{e}_A)^{\frac{1}{m}} \mid f_U \mid^2 d\mu(z) \\ &\leqq \left(\inf_U h_A(\mathbf{e}_A, \mathbf{e}_A)\right)^{-\frac{1}{m}} \end{aligned} \tag{2.3}$$

hold, where $d\mu(z)$ denotes the standard Lebesgue measure on the coordinate. Hence by the submeanvalue property of plurisubharmonic functions,

$$h_A^{\frac{1}{m}} \cdot |\,\sigma\,|^{\frac{2}{m}}\,(x) \leqq \left(\frac{h_A(\mathbf{e}_A, \mathbf{e}_A)(x)}{\inf_U h_A(\mathbf{e}_A, \mathbf{e}_A)}\right)^{\frac{1}{m}} \cdot \pi^{-n} \cdot |dz_1 \wedge \cdots \wedge dz_n|^2(x) \tag{2.4}$$

holds. Let us fix a C^∞-volume form dV on X. Since X is compact and every line bundle on a contractible Stein manifold is trivial, we have the following lemma.

Lemma 2.1 *There exists a positive constant C independent of the line bundle A and the C^∞-metric h_A such that*

$$\limsup_{m\to\infty} h_A^{\frac{1}{m}} \cdot \hat{K}_m^A \leqq C \cdot dV$$

holds on X. □

2.2 Lower estimate of $\hat{K}_m^A$

The lower estimate of $\hat{K}_m^A$ is the essential part of the proof of Theorem 1.7.

Let h_X be any C^∞-hermitian metric on K_X. Let h_0 be an AZD on K_X defined by

$$h_0 := \text{the lower envelope of } \inf\{h \mid h \text{ is a singular hermitian metric on } K_X \text{ with } \sqrt{-1}\,\Theta_h \geqq 0, h \geqq h_X\}, \tag{2.5}$$

where inf denotes the pointwise infimum. Then by the classical theorem ([L, p.26, Theorem 5]), h_0 is an AZD with minimal singularities in the sense of Definition 5.2 below.

Let us compare h_0 and $\hat{h}_{can}$. By the L^2-extension theorem ([O]), we have the following lemma.

Lemma 2.2 *There exists a positive constant C independent of m such that*

$$K(A + mK_X, h_A \cdot h_0^{m-1}) \geqq C \cdot (h_A \cdot h_0^m)^{-1} \tag{2.6}$$

holds, where $K(A + mK_X, h_A \cdot h_0^{m-1})$ is the diagonal part of Bergman kernel of $A + mK_X$ with respect to the L^2-inner product:

$$(\sigma, \sigma') := (\sqrt{-1})^{n^2} \int_X \sigma \wedge \overline{\sigma'} \cdot h_A \cdot h_0^{m-1}, \tag{2.7}$$

where we have considered σ, σ' as $A + (m-1)K_X$ valued canonical forms (see (1.16)). □

Proof of Lemma 2.2. By the extremal property of the Bergman kernel (see for example [Kr, p.46, Proposition 1.4.16]), we have that

$$K(A + mK_X, h_A \cdot h_0^{m-1})(x) = \sup\left\{|\,\sigma(x)\,|^2; \sigma \in \Gamma(X, \mathcal{O}_X(A + mK_X) \otimes \mathcal{I}(h_0^{m-1})), \|\,\sigma\,\| = 1\right\} \tag{2.8}$$

holds for every $x \in X$, where $\|\,\sigma\,\|$ denotes the norm $(\sigma, \sigma)^{\frac{1}{2}}$. Let x be a point such that h_0 is not $+\infty$ at x. Let dV be an arbitrary C^∞-volume form on X as in Section 1.2. Then by the L^2-extension theorem ([O, O-T]) and the sufficient ampleness of A (see Sections 1.2 and 5.1), we may extend any $\tau_x \in (A+mK_X)_x$ with $h_A \cdot h_0^{m-1} \cdot dV^{-1}(\tau_x, \tau_x) = 1$ to a global section $\tau \in \Gamma(X, \mathcal{O}_X(A+mK_X) \otimes \mathcal{I}(h_0^{m-1}))$ such that

$$\|\,\tau\,\| \leqq C_0, \tag{2.9}$$

where C_0 is a positive constant independent of x and m. Let C_1 be a positive constant such that

$$h_0 \geqq C_1 \cdot dV^{-1} \tag{2.10}$$

holds on X. By (2.8), we obtain the lemma by taking $C = C_0^{-1} \cdot C_1$. □

Let $\sigma \in \Gamma(X, \mathcal{O}_X(A + mK_X) \otimes \mathcal{I}(h_0^{m-1}))$ such that

$$\left| \int_X \sigma \wedge \bar{\sigma} \cdot h_A \cdot h_0^{m-1} \right| = 1 \tag{2.11}$$

and

$$| \sigma |^2 (x) = K(A + mK_X, h_A \cdot h_0^{m-1})(x) \tag{2.12}$$

hold, i.e., σ is a peak section at x. Then by the Hölder inequality we have that

$$\begin{aligned} \left| \int_X h_A^{\frac{1}{m}} \cdot (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right| &\leqq \left(\int_X h_A \cdot h_0^m \cdot | \sigma |^2 \cdot h_0^{-1} \right)^{\frac{1}{m}} \cdot \left(\int_X h_0^{-1} \right)^{\frac{m-1}{m}} \\ &\leqq \left(\int_X h_0^{-1} \right)^{\frac{m-1}{m}} \end{aligned} \tag{2.13}$$

hold. Hence we have the inequality:

$$\hat{K}_m^A(x) \geqq K(A + mK_X, h_A \cdot h_0^{m-1})(x)^{\frac{1}{m}} \cdot \left(\int_X h_0^{-1} \right)^{-\frac{m-1}{m}} \tag{2.14}$$

holds. Now we shall consider the limit:

$$\limsup_{m \to \infty} h_A^{\frac{1}{m}} \cdot K(A + mK_X, h_A \cdot h_0^{m-1})^{\frac{1}{m}}. \tag{2.15}$$

Let us recall the following result.

Lemma 2.3 *([D, p.376, Proposition 3.1]) Let M be a smooth projective variety and let H be a sufficiently ample line bundle on M and let h_H be a C^∞-hermitian metric on H with strictly positive curvature. Then for every pseudoeffective singular hermitian line bundle (L, h_L) on M,*

$$\limsup_{m \to \infty} h_H^{\frac{1}{m}} \cdot K(K_M + H + mL, h_H \cdot h_L^m)^{\frac{1}{m}} = h_L^{-1}$$

holds. □

Remark 2.4 *In ([D, p.376, Proposition 3.1], J.P. Demailly only considered the local version of Lemma 2.3. But the same proof works in our case by the sufficiently ampleness of H. More precisely if we take H to be sufficiently ample, by the L^2-extension theorem [O-T, O], there exists an interpolation operator:*

$$I_x : A^2(x, (K_M + H + L)_x, h_H \cdot h_F, \delta_x) \to A^2(M, K_M + H + L, h_A \cdot h_L)$$

such that the operator norm of I_x is bounded by a positive constant independent of x and (L, h_L), where $A^2(M, K_M + H + F, h_H \cdot h_L)$ denotes the Hilbert space defined by

$$A^2(M, K_M + H + L, h_H \cdot h_L) := \left\{ \sigma \in \Gamma(M, \mathcal{O}_M(K_M + H + L)), \left| \int_M h_H \cdot h_L \cdot \sigma \wedge \bar{\sigma} \right| < +\infty \right\}$$

with the L^2-inner product:

$$(\sigma, \sigma') := (\sqrt{-1})^{n^2} \int_M h_H \cdot h_L \cdot \sigma \wedge \overline{\sigma'} \qquad (n := \dim M)$$

and $A^2(x, (K_M + H + L)_x, h_H \cdot h_L, \delta_x)$ is defined similarly, where δ_x is the Dirac measure supported at x. This is the precise meaning of sufficiently ampleness of H in Lemma 2.3. We note that if $h_L(x) = +\infty$, then $A^2(x, (K_M + H + L)_x, h_H \cdot h_L, \delta_x) = 0$. In this setting, for a Stein local coordinate neighborhood U,

$$\limsup_{m \to \infty} h_H^{\frac{1}{m}} \cdot K(K_M + H + mL, h_H \cdot h_L^m)^{\frac{1}{m}} | U =$$

$$\limsup_{m\to\infty} h_H^{\frac{1}{m}} \cdot K(K_M + H + mL|U, h_H \cdot h_L^m|U)^{\frac{1}{m}} = h_L^{-1}|U$$

hold. This kind of localization principle of Bergman kernels is quite standard. Moreover for an arbitrary pseudoeffective singular hermitian line bundle (F, h_F) on M,

$$\limsup_{m\to\infty} (h_H \cdot h_F)^{\frac{1}{m}} \cdot K(K_M + H + F + mL, h_H \cdot h_F \cdot h_L^m)^{\frac{1}{m}} = h_L^{-1}$$

holds almost everywhere on M.

In fact the L^2-extension theorem ([O-T, O]) implies the inequality:

$$\limsup_{m\to\infty} (h_H \cdot h_F)^{\frac{1}{m}} \cdot K(K_M + H + F + mL, h_H \cdot h_F \cdot h_L^m)^{\frac{1}{m}} \geqq h_L^{-1} \tag{2.16}$$

and the converse inequality is elementary. See [D] for details and applications.

The reason why we can take H independent of (L, h_L) is the fact that the L^2-extension theorem ([O-T, O]) is uniform with respect to plurisubharmonic weights. Moreover the extension norm is independent of the weights. $\square$

We may and do assume that A is sufficiently ample in the sense of Lemma 2.3. Anyway to define $\hat{h}_{can}$ we will replace A by ℓA and take the upper limit as ℓ tends to infinity. Then by Lemma 2.3 letting m tend to infinity in (2.14), we have the following lemma.

Lemma 2.5

$$\limsup_{m\to\infty} h_A^{\frac{1}{m}} \cdot \hat{K}_m^A \geqq \left(\int_X h_0^{-1}\right)^{-1} \cdot h_0^{-1}$$

holds. $\square$

By Lemmas 2.1 and 2.5, we see that

$$\hat{K}_\infty^A := \limsup_{m\to\infty} h_A^{\frac{1}{m}} \cdot \hat{K}_m^A \tag{2.17}$$

exists as a bounded semipositive (n, n)-form on X $(n = \dim X)$. We set

$$\hat{h}_{can,A} := \text{the lower envelope of } (K_\infty^A)^{-1}. \tag{2.18}$$

2.3 Independence of $\hat{h}_{can,A}$ from h_A

In the above construction, $\hat{h}_{can,A}$ depends on the choice of the C^∞-hermitian metric h_A apriori. But we have the following lemma.

Lemma 2.6 *$\hat{K}_\infty^A = \limsup_{m\to\infty} h_A^{\frac{1}{m}} \cdot \hat{K}_m^A$ is independent of the choice of the C^∞-hermitian metric h_A. Hence $h_{can,A}$ is independent of the choice of the C^∞-hermitian metric h_A.* $\square$

Proof of Lemma 2.6. Let h'_A be another C^∞-hermitian metric on A. We define for $x \in X$

$$(\hat{K}_m^A)'(x) := \sup\left\{ |\,\sigma\,|^{\frac{2}{m}}(x);\, \sigma \in \Gamma(X, \mathcal{O}_X(A + mK_X)), \left|\int_X (h'_A)^{\frac{1}{m}} \cdot (\sigma \wedge \bar{\sigma})^{\frac{1}{m}}\right| = 1 \right\}. \tag{2.19}$$

We note that the ratio h_A/h'_A is a positive C^∞-function on X and

$$\lim_{m\to\infty} \left(\frac{h_A}{h'_A}\right)^{\frac{1}{m}} = 1 \tag{2.20}$$

uniformly on X. Since the definitions of $\hat{K}_m^A$ and $(\hat{K}_m^A)'$ use the extremal properties, we see easily that for every positive number ε, there exists a positive integer N such that for every $m \geqq N$

$$(1 - \varepsilon)(\hat{K}_m^A)' \leqq \hat{K}_m^A \leqq (1 + \varepsilon)(\hat{K}_m^A)' \tag{2.21}$$

holds on X. This completes the proof of Lemma 2.6. $\square$

2.4 Completion of the proof of Theorem 1.7

By Lemma 2.5 we have the following lemma.

Lemma 2.7 *$\hat{h}_{can,A}$ is an AZD with minimal singularities on K_X.* $\square$

Proof. Let h_0 be an AZD of K_X with minimal singularities (cf. Definition 5.2) constructed as in (2.5). Then by Lemma 2.5 we see that

$$\hat{h}_{can,A} \leqq \left(\int_X h_0^{-1}\right) \cdot h_0 \tag{2.22}$$

holds. Hence we see

$$\mathcal{I}(\hat{h}^m_{can,A}) \supseteq \mathcal{I}(h_0^m) \tag{2.23}$$

holds for every $m \geqq 1$. This implies that

$$H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(h_0^m)) \subseteq H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(\hat{h}^m_{can,A})) \subseteq H^0(X, \mathcal{O}_X(mK_X))$$

hold, hence

$$H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(\hat{h}^m_{can,A})) \simeq H^0(X, \mathcal{O}_X(mK_X))$$

holds for every $m \geqq 1$. And by the construction and the classical theorem of Lelong ([L, p.26, Theorem 5]) stated in Section 1.3, $\hat{h}_{can,A}$ has semipositive curvature current. Hence $\hat{h}_{can,A}$ is an AZD on K_X and depends only on X and A by Lemma 2.6. We note that $\hat{h}_{can,A}$ is less singular than h_0 by (2.22). Since h_0 has minimal singularities, $\hat{h}_{can,A}$ has minimal singularities, too. $\square$

Let us consider

$$\hat{K}_\infty := \sup_A \hat{K}_{\infty,A}, \tag{2.24}$$

where sup means the pointwise supremum and A runs all the sufficiently ample line bundle on X. Then by Lemma 2.1, we see that $\hat{K}_\infty$ is a well defined semipositive (n,n)-form on X. We set

$$\hat{h}_{can} := \text{the lower envelope of } \hat{K}_\infty^{-1}. \tag{2.25}$$

Then by the construction, $\hat{h}_{can} \leqq \hat{h}_{can,A}$ for every ample line bundle A. Since $\hat{h}_{can,A}$ is an AZD on K_X, $\hat{h}_{can}$ is also an AZD on K_X indeed (again by [L, p.26, Theorem 5]). Since $\hat{h}_{can,A}$ depends only on X and A, $\hat{h}_{can}$ is uniquely determined by X. By Lemma 2.5, it is clear that $\hat{h}_{can}$ is an AZD on K_X with minimal singularities in the sense of Definition 5.2 below. This completes the proof of Theorem 1.7. $\square$

Remark 2.8 *As in Section 2.2, we see that $\hat{h}_{can}$ is an AZD on K_X with minimal singularities (cf. Definition 5.2).* $\square$

2.5 Comparison of h_{can} and $\hat{h}_{can}$

Suppose that X has nonnegative Kodaira dimension. Then by Theorem 1.3, we can define the canonical AZD h_{can} on K_X. We shall compare h_{can} and $\hat{h}_{can}$.

Theorem 2.9 *$\hat{h}_{can,A} \leqq h_{can}$ holds on X. In particular $\hat{h}_{can} \leqq h_{can}$ holds on X* $\square$

Proof of Theorem 2.9. If X has negative Kodaira dimension, then the right hand side is infinity. Hence the inequality is trivial. Suppose that X has nonnegative Kodaira dimension. Let $\sigma \in \Gamma(X, \mathcal{O}_X(mK_X))$ be an element such that

$$\left| \int_X (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right| = 1. \tag{2.26}$$

Let $x \in X$ be an arbitrary point on X. Since $\mathcal{O}_X(A)$ is globally generated by the definition of A, there exists an element $\tau \in \Gamma(X, \mathcal{O}_X(A))$ such that $\tau(x) \neq 0$ and $h_A(\tau, \tau) \leqq 1$ on X. Then we see that

$$\left| \int_X h_A(\tau, \tau)^{\frac{1}{m}} \cdot (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right| \leqq 1 \tag{2.27}$$

holds. This implies that

$$\hat{K}^A_m(x) \geqq |\tau(x)|^{\frac{2}{m}} \cdot K_m(x) \tag{2.28}$$

holds at x. Noting $\tau(x) \neq 0$, letting m tend to infinity, we see that

$$\hat{K}^A_\infty(x) \geqq K_\infty(x) \tag{2.29}$$

holds. Since x is arbitrary, this completes the proof of Theorem 2.9. □

Remark 2.10 *The equality $h_{can} = \hat{h}_{can}$ implies the abundance of K_X, if the numerical dimension of $(K_X, \hat{h}_{can})$ is equal to the numerical dimension of K_X. This problem will be treated in [T10].* □

By the same proof we obtain the following comparison theorem (without assuming X has nonnegative Kodaira dimension).

Theorem 2.11 *Let A, B a sufficiently ample line bundle on X. Suppose that $B-A$ is globally generated, then*

$$\hat{h}_{can,B} \leqq \hat{h}_{can,A}$$

holds. □

Remark 2.12 *Theorem 2.11 implies that*

$$\hat{h}_{can} = \lim_{\ell \to \infty} \hat{h}_{can,\ell A} \tag{2.30}$$

holds for any ample line bundle A on X. □

Remark 2.13 *By Kodaira's lemma and Theorem 2.11, we see that $\hat{h}_{can,A}$ is independent of A when K_X is big. But it is not clear wheter $\hat{h}_{can,A}$ is independent of A, when K_X is pseudoeffective but not big. But by Lemma 2.5, one can easily deduce that for any two members of $\{\hat{h}_{can,A}\}$, the ratio of these metrics is uniformly bounded on X, where A runs all the ample line bundles on X.* □

2.6 Canonical volume forms on open manifolds

The construction in Section 1.1 can be generalized to an arbitrary complex manifold. This is just a formal generalization. But it arises the many interesting problems and also is important to consider the degeneration, This subsection is not essential in the later argument. Hence one may skip it. Let M be a complex manifold. For every positive integer m, we set

$$Z_m := \left\{ \sigma \in \Gamma(M, \mathcal{O}_M(mK_M));\, \left| \int_M (\sigma \wedge \overline{\sigma})^{\frac{1}{m}} \right| < +\infty \right\}$$

and

$$K_{M,m} := \sup \left\{ |\sigma|^{\frac{2}{m}}\, ;\ \sigma \in \Gamma(M, \mathcal{O}_M(mK_M)),\, \left| \int_M (\sigma \wedge \overline{\sigma})^{\frac{1}{m}} \right| \leqq 1 \right\},$$

where sup denotes the pointwise supremum.

Proposition 2.14

$$K_{M,\infty} := \limsup_{m \to \infty} K_{M,m}$$

exists and if $Z_m \neq 0$ for some $m > 0$, then $K_{M,\infty}$ is not identically 0 and

$$h_{can,M} := \textit{the lower envelope of } \frac{1}{K_{M,\infty}}$$

is a well defined singular hermitian metric on K_M with semipositive curvature current. □

By definition, $h_{can,M}$ is invariant under the automorphism group $\mathrm{Aut}(M)$. Hence we obtain the following:

Proposition 2.15 *Let Ω be a homogeneous bounded domain in $\mathbb{C}^n$. Then $h^{-1}_{can,\Omega}$ is a constant multiple of the Bergman volume form on Ω.* $\square$

For a general bouded domain in $\mathbb{C}^n$ it seems to be very difficult to calculate the invariant volume form h^{-1}_{can}. Let us consider the punctured disk

$$\Delta^* := \{t \in \mathbb{C} |\ 0 < |t| < 1\}.$$

Then one sees that unlike the Bergman kernel, h_{can,Δ^*} reflects the puncture. The following conjecture seems to be very plausible. But at this moment I do not know how to solve.

Conjecture 2.16

$$h^{-1}_{can,\Delta^*} = O\left(\frac{\sqrt{-1}dt \wedge d\bar{t}}{|t|^2 (\log |t|)^2}\right)$$

holds. $\square$

Conjecture 2.16 is very important in many senses (see Proposition 2.22 below and Remark 4.16 for example). In particular it seems to be the key to extend Theorem 4.2 to LC pairs.

Next we shall consider the following situation. Let X be a smooth projective variety and let D be a divisor with simple normal crossings on X. We set $M := X \backslash D$. Let A be a sufficiently ample line bundle on X in the sense of Proposition 5.1. Let h_A be a C^∞-hermitian metric on A with strictly positive curvature. We define

$$\hat{K}^A_m := \sup\{\, |\sigma|^{\frac{2}{m}};\ \sigma \in \Gamma(M, \mathcal{O}_M(A + mK_M)),\ \parallel \sigma \parallel_{\frac{1}{m}} = 1\}, \tag{2.31}$$

where

$$\parallel \sigma \parallel_{\frac{1}{m}} := \left| \int_X h_A^{\frac{1}{m}} \cdot (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right|^{\frac{m}{2}}. \tag{2.32}$$

And as (1.7) we define

$$\hat{h}_{can,A} := \text{the lower envelope of } \liminf_{m\to\infty} (\hat{K}^A_m)^{-1}. \tag{2.33}$$

As Lemma 2.6, we see that $\hat{h}_{can,A}$ is independent of the choice of h_A. We set

$$\hat{h}_{can,M} := \text{the lower envelope of } \inf_A \hat{h}_{can,A},$$

where A runs all the ample line bundle on X. We note that

$$\{\, \sigma \in \Gamma(M, \mathcal{O}_M(A + mK_M)),\ \parallel \sigma \parallel_{\frac{1}{m}} < \infty\} \simeq \Gamma(X, \mathcal{O}_X(mK_X + (m-1)D))$$

holds by a simple calculation.

Definition 2.17 *Let X be a smooth projective variety and let D be a divisor with simple normal crossings on X. Let σ_D be a global holomorphic section of $\mathcal{O}_X(D)$ with divisor D. $M := X\backslash D$ is said to be of finite volume, if there exists an AZD h of $K_X + D$ such that*

$$\int_M h^{-1} \cdot h_D$$

is finite, where $h_D := |\sigma_D|^{-2}$. $\square$

Remark 2.18 *In the above definition, h is not an AZD of minimal singularities (cf. Definition 5.2), when $K_X + D$ is ample.* $\square$

Example 2.19 *Let ω_E be a complete Kähler-Einstein form on M such that $-Ric_{\omega_E} = \omega_E$ ([Ko]). We set $n := \dim X$. Then $h = (\omega_E^n) \cdot h_D$ is an AZD on $K_X + D$ such that*

$$\int_M h^{-1} \cdot h_D = \int_M \omega_E^n < +\infty.$$

Hence M is of finite volume. $\square$

Theorem 2.20 *Let X be a smooth projective variety and let D be a divisor with simple normal crossings on X. We set $M := X \backslash D$. Suppose that M is of finite volume. Then $h_{can,M} \cdot h_D$ is an AZD on $K_X + D$.* $\square$

Proof. Let h_0 be an AZD on $K_X + D$ such that $(M, h_0^{-1} \cdot h_D)$ is of finite volume. Then by Hölder's inequality, for every $\sigma \in \Gamma(X, \mathcal{O}_X(mK_X + (m-1)D))$,

$$\left| \int_M h_A^{\frac{1}{m}} \cdot h_D \cdot (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right| \leqq \left(\int_M h_A \cdot h_0^m \cdot \mid \sigma \mid^2 \cdot h_0^{-1} \cdot h_D \right)^{\frac{1}{m}} \cdot \left(\int_M h_0^{-1} \cdot h_D \right)^{\frac{m-1}{m}} \tag{2.34}$$

holds. By the assumption, we know that the second factor on the right-hand side is finite. We shall show the fisrt factor in the right-hand side is finite. We note that for a local generator τ of $K_X + D$ on an open set U, $\log h_0(\tau, \tau)$ is locally bounded from below on U, since $\log h_0(\tau, \tau)$ is plurisuperharmonic. Let σ_D be as in Definition 2.17. Then $\parallel \sigma_D \parallel^2 \cdot (h_0^{-1} \cdot h_D)$ is equal to $(|\tau|^2 \cdot h_0(\tau, \tau)^{-1}) \cdot (\parallel \sigma_D \parallel^2 \cdot h_D)$ and is a bounded volume form on X, where $\parallel \sigma_D \parallel$ denotes the hermitian norm of σ_D with respect to a fixed C^∞-hermitian metric on $\mathcal{O}_X(D)$. Then since σ belongs to $\Gamma(X, \mathcal{O}_X(A + mK_X + (m-1)D))$,

$$\int_M h_A \cdot h_0^m \cdot \mid \sigma \mid^2 \cdot h_0^{-1} \cdot h_D < \infty$$

holds. Hence by (2.34), we have the inequality:

$$\hat{K}_m^A(x) \geqq K(A + mK_M, h_A \cdot h_0^{m-1} \cdot h_D^{m-1})(x)^{\frac{1}{m}} \cdot \left(\int_M h_0^{-1} \cdot h_D \right)^{-\frac{m-1}{m}} \tag{2.35}$$

holds. Since $(M, h_0^{-1} \cdot h_D)$ has finite volume, by the same argument as in Section 2.2, letting m tend to infinity, by Lemma 2.3 and Remark 2.4. , we see that

$$\hat{h}_{can,A} \leqq h_0 \cdot h_D^{-1} \cdot \left(\int_M h_0^{-1} \cdot h_D \right) \tag{2.36}$$

holds.

On the other hand, we obtain the upper estimate of $\hat{K}_m^A$ as follows. Let dV be a C^∞-volume form on X. By the submeanvalue inequality for plurisubharmonic functions as in Section 2.1, we see that there exists a positive number C' independent of m such that

$$h_A^{\frac{1}{m}} \cdot \hat{K}_m^A \leqq C' \cdot \frac{dV}{\parallel \sigma_D \parallel^2}$$

holds on X. Hence $\hat{h}_{can,A}$ exists as a well defined singular hermitian metric on $K_X + D$ and by the construction $\hat{h}_{can,A}$ has semipositive curvature current. By (2.36) $\hat{h}_{can,A} \cdot h_D$ is an AZD on $K_X + D$. This implies that $\hat{h}_{can,M} \cdot h_D$ is an AZD on $K_X + D$. This completes the proof of Theorem 2.20. $\square$

The following problem seems to be interesting.

Problem 2.21 *Let X be a smooth projective variety and let D be a divisor with only normal crossings on X such that $K_X + D$ is ample. We set $M := X \backslash D$. Is $\hat{h}_{can,M}^{-1}$ a constant multiple of the Kähler-Einstein volume form on X constructed in [Ko] ?* $\square$

If the above problem is affimative $(M, \hat{h}^{-1}_{can,M})$ is of finite volume. The following is the first step to solve the problem.

Proposition 2.22 *Let (X, D), $\hat{h}_{can}$ be as in Problem 2.21. If Conjecture 2.16 holds, then*

$$\int_M \hat{h}^{-1}_{can,M} < \infty$$

holds. □

Proof. Let $(U, t_1, \cdots, t_n)$ be a local coordinate such that

1. U is biholomorphic to Δ^n by $(t_1, \cdots, t_n)$,
2. $U \cap D = \{(t_1, \cdots, t_n) \in \Delta^n | t_1 \cdots t_k = 0\}$ holds for some k.

We note that the equality:

$$\hat{h}_{can,U\setminus D} = h_{can,U\setminus D}$$

holds as Lemma 2.6, since $A|U$ is trivial with smooth metric. For every subset V of M, we see that the monotonicity:

$$\hat{h}^{-1}_{can,M} \leqq \hat{h}^{-1}_{can,V}$$

holds by the above construction. Hence we see that

$$\hat{h}^{-1}_{can,M} \leqq \hat{h}^{-1}_{can,U\setminus D}$$

holds. Then by Conjecture 2.16, we see that $\hat{h}_{can,M}$ is of locally of finite volume at every point on $U \cap D$. This completes the proof. □

3 Variation of $\hat{h}_{can}$ under projective deformations

In this section we shall prove Theorem 1.13. The main ingredient of the proof is the plurisubharmonic variation property of Bergman kernels ([Ber, B-P, T3]).

3.1 Construction of $\hat{h}_{can}$ on a family

Let $f : X \longrightarrow S$ be a proper surjective projective morphism with connected fibers between complex manifolds as in Theorem 1.13.

The construction of $\hat{h}_{can}$ can be performed simultaeneously on the family as follows. The same construction works for flat projective family with only canonical singularities. But for simplicity we shall work on smooth category.

Let S° be the maximal nonempty Zariski open subset of S such that f is smooth over S° and let us set $X^\circ := f^{-1}(S^\circ)$.

Hereafter we shall assume that $\dim S = 1$. The general case of Theorem 1.13 easily follows from just by cutting down S to curves (cf. Section 3.3 below). Let A be a sufficiently ample line bundle on X such that for every pseudoeffective singular hermitian line bundle (L, h_L), $\mathcal{O}_X(A + L) \otimes \mathcal{I}(h_L)$ and $\mathcal{O}_X(K_X + A + L) \otimes \mathcal{I}(h_L)$ are globally generated and $\mathcal{O}_{X_s}(A + L|X_s) \otimes \mathcal{I}(h_L|X_s)$ and $\mathcal{O}_{X_s}(K_{X_s} + A + L|X_s) \otimes \mathcal{I}(h_L|X_s)$ are globally generated for every $s \in S^\circ$ as long as $h_L|X_s$ is well defined (cf. Proposition 5.1). Let h_A be a C^∞-hermitian metric on A with strictly positive curvature. We set

$$E_m := f_*\mathcal{O}_X(A + mK_{X/S}). \tag{3.1}$$

Since we have assumed that $\dim S = 1$, E_m is a vector bundle on S for every $m \geqq 1$. We denote the fiber of the vector bundle over $s \in S$ by $E_{m,s}$. Then we shall define the sequence of $\frac{1}{m}A$-valued relative volume forms by

$$\hat{K}^A_{m,s} := \sup\left\{ |\sigma|^{\frac{2}{m}} ; \sigma \in E_{m,s}, \left| \int_{X_s} h_A^{\frac{1}{m}} \cdot (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right| = 1 \right\} \tag{3.2}$$

for every $s \in S^\circ$, where sup denotes the pointwise supremum. This fiberwise construction is different from that in Section 1.2 at the point that we use $E_{m,s}$ instead of $\Gamma(X_s, \mathcal{O}_{X_s}(A|X_s + mK_{X_s}))$. We note that the difference occurs only over at most countable union of proper analytic subsets in S° by the upper-semicontinuity theorem of cohomologies.

We define the relative $|A|^{\frac{2}{m}}$ valued volume form $\hat{K}^A_m$ by

$$\hat{K}^A_m|X_s := \hat{K}^A_{m,s} \quad (s \in S^\circ) \tag{3.3}$$

and a relative volume form $\hat{K}^A_\infty$ by

$$\hat{K}^A_\infty|X_s := \limsup_{m\to\infty} h_A^{\frac{1}{m}} \cdot \hat{K}^A_{m,s} \quad (s \in S^\circ). \tag{3.4}$$

We define a singular hermitian metrics on $\frac{1}{m}A + K_{X/S}$ by

$$\hat{h}_{m,A} := \text{the lower envelope of } (\hat{K}^A_m)^{-1}. \tag{3.5}$$

We set

$$\hat{h}_{can,A} := \text{the lower envelope of } \liminf_{m\to\infty} h_A^{-\frac{1}{m}} \cdot \hat{h}_{m,A}. \tag{3.6}$$

Then we define

$$\hat{h}_{can} := \text{the lower envelope of } \inf_A \hat{h}_{can,A}, \tag{3.7}$$

where A runs all the ample line bundles on X. At this moment, $\hat{h}_{can}$ is defined only on $K_{X/S}|X^\circ$. The extension of $\hat{h}_{can}$ to the singular hermitian metric on the whole $K_{X/S}$ will be discussed later.

3.2 Semipositivity of the curvature current of $\hat{h}_{m,A}$

Let $E_{m,s}$ denote the fiber of the vector bundle E_m at s. For $s \in S^\circ$, we define the pseudonorm $\| \sigma \|_{\frac{1}{m}}$ of $\sigma \in E_{m,s}$ by

$$\| \sigma \|_{\frac{1}{m}} := \left| \int_{X_s} h_A^{\frac{1}{m}} \cdot (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right|^{\frac{m}{2}}. \tag{3.8}$$

Now we quote the following crucial result. This is a generalization of [Ka1, p.57,Theorem 1].

Theorem 3.1 *([B-P, Corollary 4.2]) Let $p : X \to Y$ be a smooth projective fibraition and let (L, h_L) be a pseudoeffective singular hermitian line bundle on X. Let m be a positive integer. Suppose that $E := p_*\mathcal{O}_X(mK_{X/Y} + L)$ is locally free. We set*

$$K^L_m(x) := \sup\left\{ |\sigma|^{\frac{2}{m}}(x)\,;\, \sigma \in E_{p(x)}, \left| \int_{X_{p(x)}} h_L \cdot (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right| = 1 \right\}.$$

Then $h_{NS} := (K^L_m)^{-m}$ is a singular hermitian metric on $mK_{X/Y} + L$ with semipositive curvature current. $\square$

Remark 3.2 *In [B-P, Corollary 4.2], they have assumed that for every $s \in S^\circ$, every global holomorphic section of $(mK_{X/Y} + L)|X_s$ extends locally to a holomorphic section of $mK_{X/Y} + L$ on a neighborhood of X_s. Apparently they have misunderstood that this extension property is equivalent to the local freeness of the direct image $E = p_*\mathcal{O}_X(mK_{X/Y} + L)$. Actually without assuming such an extension property, the local freeness of E is automatic in the case of* $\dim Y = 1$, *since the direct image E is always torsion free. In fact for $y \in Y$ the fiber E_y of the vector bundle E at y is a subspace of $H^0(X_y, \mathcal{O}_{X_y}(mK_{X_y} + L|X_y))$*

such that every element of E_y is locally holomorphically extendable on a neighborhood of X_y. Hence the proof of [B-P, Corollary 4.2] is valid in this case and Theorem 3.1 is valid as it stands.

Or we can argue as follows. By the upper-semicontinuity of $h^0(X_y, \mathcal{O}_{X_y}(mK_{X_y}+L|X_y))$, there exists a nonempty Zariski open subset Y_0 of Y such that for every $y \in Y_0$, every element of $H^0(X_y, \mathcal{O}_{X_y}(mK_{X_y}+L|X_y))$ extends on a neighborhood of X_y as a holomorphic section of $mK_{X/Y}+L$. We note the h_L dominates a C^∞-metric of L by the assumption and $L^{2/m}$-pseudonorm is lower-semicontinuous on E because h_L is lower-semicontinuous. Hence $K^L_m(x)$ is locally bounded from above as a section of the real line bundle $|K_{X/Y}|^2 \otimes |L|^{\frac{2}{m}}$ over X. Let U be an open subset of X such that $mK_{X/Y}+L$ has holomorphic frame $\mathbf{e}$ on U. Then $\log(K^L_m/|\mathbf{e}|^{2/m})$ is plurisubharmonic on $U \cap f^{-1}(Y_0)$ by [B-P, Corollary 4.2]. Then since $\log(K^L_m/|\mathbf{e}|^{2/m})$ is locally bounded from above, we may apply the classical extension theorem for plurisubharmonic functions ([H-P, p.704, Theorem 1.2 (b)]). Hence we may extend h_{NS} as a singular hermitian metric with semipositive curvature on the whole $mK_{X/Y}+L$. This argument is better, since we do not really use the local freeness of E. Hence Theorem 3.1 holds without assuming the local freeness of E. □

Theorem 3.1 immediately implies the semipositivity of the curvature current of $\hat{h}_{can,A}$.

Corollary 3.3 *$\hat{h}_{m.A}$ has semipositive curvature current on X°.* □

Now let us consider the behavior of $\hat{h}_{m,A}$ along $X \backslash X^\circ$. Let p be a point in $S \backslash S^\circ$. Since the problem is local, we may and do assume S is the unit open disk Δ in $\mathbb{C}$ with center 0 for the time being and p is the origin 0.

The following argument is taken from [F, p. 782,Lemma (1,11)]. Let $\sigma \in \Gamma(X, \mathcal{O}_X(A + mK_{X/S}))$ be a section such that $\sigma|X_0 \neq 0$. We consider the (multivalued) $\frac{1}{m}A$-valued relative canonical form:

$$\eta := \sigma^{\frac{1}{m}}. \tag{3.9}$$

We may and do assume that the support of the fiber X_0 is a divisor with simple normal crossings. Let

$$X_0 = \sum_i \nu_i X_{0,i} \tag{3.10}$$

be the irreducible decomposition. We note that the (multivalued) $\frac{1}{m}A$-valued canonical form

$$f^*dt \wedge \eta \tag{3.11}$$

does not vanish identically on X_0 by the assumption: $\sigma|X_0 \neq 0$. Since the zero divisor of f^*dt is $\sum_i(\nu_i - 1)X_{i,0}$, we see that for some i, $\eta|X_{i,0}$ is a nonzero $\frac{1}{m}A$-valued meromorphic canonical form. Hence

$$\| \sigma \|_{\frac{1}{m}} := \left| \int_{X_s} h_A^{\frac{1}{m}} (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right|^{\frac{m}{2}} \tag{3.12}$$

has a positive lower bound around $p = 0 \in S$. This implies that $h_{m,A}$ has positive lower bound around X_0. Then we see that (fixing a local holomorphic frame of $A + mK_{X/S}$) $-\log h_{m,A}$ is locally bounded from above around X_0 and extends across X_0 as a (local) plurisubhramonic function by [H-P, p.704, Theorem 1.2 (b)]. This implies that $\hat{h}_{m,A}$ is bounded from below by a smooth metric along the boundary $X \setminus X^\circ$. Hence $\hat{h}_{m,A}$ extends to a singular hermitian metric of $\frac{1}{m}A + K_{X/S}$ with semipositive curvature on the whole X by the same manner as above. Now we set

$$\hat{h}_{can,A} := \text{the lower envelope of } \liminf_{m\to\infty} h_A^{-\frac{1}{m}} \cdot \hat{h}_{m,A}. \tag{3.13}$$

To extend $\hat{h}_{can,A}$ across $X \setminus X^\circ$, we use the following useful lemma.

Lemma 3.4 *([B-T, Corollary 7.3]) Let $\{u_j\}$ be a sequence of plurisubharmonic functions locally bounded above on the bounded open set Ω in $\mathbb{C}^n$. Suppose further*

$$\limsup_{j\to\infty} u_j$$

is not identically $-\infty$ on any component of Ω. Then there exists a plurisubharmonic function u on Ω such that the set of points:

$$\{x \in \Omega \mid u(x) \neq (\limsup_{j\to\infty} u_j)(x)\}$$

is pluripolar. $\square$

Since $\hat{h}_{m,A}$ extends to a singular hermitian metric on $\frac{1}{m}A+K_{X/S}$ with semipositive curvature current on the whole X and

$$\hat{h}_{can,A} := \text{the lower envelope of } \liminf_{m\to\infty} h_A^{-\frac{1}{m}} \cdot \hat{h}_{m,A} \tag{3.14}$$

exists as a singular hermitian metric on $K_{X/S}$ on $X^\circ = f^{-1}(S^\circ)$, we see that $\hat{h}_{can,A}$ extends to a singular hermitian metric on the whole X with semipositive curvature current by Lemma 3.4.

Repeating the same argument we see that $\hat{h}_{can}$ is a well defined singular hermitian metric on $K_{X/S} \mid X^\circ$ with semipositive curvature current and it extends to a singular hermitian metric on $K_{X/S}$ with semipositive curvature current on the whole X.

3.3 Case $\dim S > 1$

In Sections 3.1,3.2, we have assumed that $\dim S = 1$. In this subsection, we shall extend $\hat{h}_{can}$ as a singular hermitian metric on $K_{X/S}$ over X with semipositive curvature in the case of $\dim S > 1$. The proof is done just by slicing, i.e., we slice the base S by families of curves and apply classical extension theorems for plurisubharmonic functions or closed semipositve currents. Let us assume that $\dim S > 1$ holds. In this case $E_m = f_*\mathcal{O}_X(A + mK_{X/S})$ may not be locally free on S°. If E_m is not locally free at $s_0 \in S^\circ$, then $\hat{K}^A_\infty$ may not be well defined or may be discontinuous at s_0, because in this case the fiber E_{m,s_0} is defined as a maximal linear subspace of $\Gamma(X_{s_0}, \mathcal{O}_{X_{s_0}}(A|X_{s_0} + mK_{X_{s_0}}))$ such that every element of the subspace is extendable to a holomorphic section of $A + mK_{X/S}$ on a neighborhood of X_{s_0}. See (3.28) below. We set for $m \geqq 1$.

$$V_m := \{s \in S^\circ \mid E_m \text{ is not locally free at } s\}. \tag{3.15}$$

Then V_m is of codimension ≥ 2, since E_m is torsion free. By the construction, apriori $\hat{h}_{can}$ is well defined only on $S^\circ \setminus \cup_{m=1}^\infty V_m$. We note that apriori $\hat{h}_{m,A}$ defined only on $f^{-1}(S^\circ \setminus V_m)$. Then since $f^{-1}(V_m)$ is of codimension $\geqq 2$ in X°, by the Hartogs type extension [H, p.71, Theorem 6], we may extend $\hat{h}_{m,A}$ across $f^{-1}(V_m)$ as a singular hermitian metric of $\frac{1}{m}A + K_{X/S}|X^\circ$ with semipositive curvature current. Or more directly, one may use the argument in Remark3.2 to extend $\hat{h}_{m,A}$ across $f^{-1}(V_m)$. The extension theorem [H, p.71, Theorem 6] is stated for closed semipositive $(1,1)$ currents. In our case, we need the extension of plurisubharmonic functions. But these two extensions are obviously related by $\partial\bar{\partial}$-Poincaré lemma (and the Hartogs extension for pluriharmonic functions). Hence by the costruction, $\hat{h}_{can}$ is extended to X° as a singular hermitian metric on $K_{X/S}|X^\circ$.

Next we shall extend $\hat{h}_{can}$ across $X \setminus X^\circ$. We note that the problem is local and birationally invariant (because the pushforwad of a closed semipositive current is again a closed semipositive). Hence by taking a suitable modification of $f : X \to S$, we may assume the followings:

(1) S is the unit open polydisk: $\Delta^k := \{(s_1, \cdots, s_k) \in \mathbb{C}^k; |s_i| < 1, i = 1, \cdots, k\}(k = \dim S > 1)$.

(2) $D := S \setminus S^\circ$ is a divisor with normal crossings on S.

Let C be a smooth irreducible curve in S satisfying:

(C1) $C \cap S^\circ \neq \emptyset$,

(C2) $f^{-1}(C)$ is smooth.

Then by the adjunction formula, we see that

$$K_{X/S}|f^{-1}(C) = K_{f^{-1}(C)/C} \tag{3.16}$$

holds. For such a curve C, noting (3.16), we may extend $\hat{h}_{can}|f^{-1}(C)\cap X^{\circ}$ to a singular hermitian metric on $K_{X/S}|f^{-1}(C)$ with semipositive curvature by the case of $\dim S = 1$.

First we shall assume that $f : X \to S$ is flat and D is smooth. In this case we may assume that

$$D = \{s_1 = 0\} \tag{3.17}$$

holds without loss of generality. We set

$$C(d_2, \cdots, d_k) := \{(s_1, \cdots, s_k) \in \Delta^k | s_2 = d_2, \cdots, s_k = d_k\}. \tag{3.18}$$

By Bertini's theorem, for a general $(d_2, \cdots, d_k) \in \Delta^{k-1}$ (here "general" means outside of a proper analytic subset), $C(d_2, \cdots, d_k)$ satisfies the above conditions (C1) and (C2) and $\{f^{-1}(C(d_2, \cdots, d_k))|(d_2, \cdots, d_k) \in \Delta^{k-1}\}$ is a flat family over Δ^{k-1} . Let (s_1) denote the divisor of s_1 and let

$$f^*(s_1) = \sum \nu_i X_i \tag{3.19}$$

be the irreducible decomposition. Let $x \in X_{i,reg}\backslash(\cup_{j\neq i} X_j)$ be a general (here "general" means outside of some proper algebraic subset) point such that there exists a member C in $\{C(d_2, \cdots, d_k)|(d_2, \cdots, d_k) \in \Delta^{k-1}\}$ such that

(1) C satisfies (C1),(C2),

(2) $f^{-1}(C)$ intersects $X_{i,reg}$ transversally at x.

Then (a branch of) $f^*s_1^{1/\nu_i}$ is a local defining function of X_i on a neighborhood W of x. And if we take W sufficiently small, we may find holomorphic functions $z_1, \cdots, z_n (n = \dim X - \dim S)$ on W such that

$$(f^*s_1^{1/\nu_i}, f^*s_2, \cdots, f^*s_k, z_1, \cdots, z_n) \tag{3.20}$$

is a local coordinate on W. Since $\hat{h}_{can}|f^{-1}(C(d_2, \cdots, d_k)\cap S^{\circ})\cap W$ extends to a singular hemitian metric on the whole slice $f^{-1}(C(d_2, \cdots, d_k))\cap W$ for every $(d_2, \cdots, d_k) \in \Delta^{k-1}$, we see that by [H-P, p.710, Theorem2.1 (c)], $\hat{h}_{can}$ extends to a singular hermitian metric with semipositive curvature current on W. In this way we see that $\hat{h}_{can}$ extends to a singular hermitian metric across a nonempty Zariski open subset of X_i for every i. Then by [H, p.71, Theorem 6], we may extend $\hat{h}_{can}$ across the whole $\sum_i X_i$. Hence in this case we may extend $\hat{h}_{can}$ across the boundary $f^{-1}(D)$.

If $D = S\backslash S^{\circ}$ is reducible and $f : X \to S$ is flat, we extend $\hat{h}_{can}$ across $f^{-1}(D_{reg})$ as above and then by [H, p.71, Theorem 6] we extend $\hat{h}_{can}$ across $f^{-1}(D_{sing})$ which is of codimension $\geqq 2$ in X thanks to the flatness of f.

If $f : X \to S$ is not flat, we shall take a flattening $\hat{f} : \hat{X} \to \hat{S}$ of $f : X \to S$ (cf. [Hiro]). In this case $\hat{X}$ and $\hat{S}$ may be singular, but we may and do take them to be normal. Let C be a curve on $\hat{S}_{reg}$ such that $\hat{f}^{-1}(C)\cap \hat{X}_{reg}$ is smooth and $C \cap \hat{S}^{\circ}_{reg} \neq \emptyset$. Although $\hat{f}^{-1}(C)$ may be singular, taking a resolution of $\hat{f}^{-1}(C)$, by the adjunction formula and the proof in the case of $\dim S = 1$, we may extend $\hat{h}_{can}|f^{-1}(C\cap \hat{S}^{\circ}_{reg})$ to a singular hermitian metric on $K_{\hat{X}_{reg}/\hat{S}_{reg}}|f^{-1}(C)\cap \hat{X}_{reg}$ with semipositive curvature. Hence by the above argument, we see that $\hat{h}_{can}$ is a well defined singular hermitian metric (with semipositive curvature current) on $K_{\hat{X}_{reg}/\hat{S}_{reg}}$ over $\hat{X}_{reg}\cap \hat{f}^{-1}(\hat{S}_{reg})$. Here we have abused the same notation $\hat{h}_{can}$ for the metric on the different space. But the metric $\hat{h}_{can}$ is birationally invariant.

Let Z be image of $\hat{X}_{sing}\cup \hat{f}^{-1}(\hat{S}_{sing})$ by the natural morphism $\hat{X} \to X$. Then Z is of codimension at least 2 in X. Then the above argument in the flat case, $\hat{h}_{can}$ extends to a singular hermitian metric on $K_{X/S}|X\backslash Z$ with semipositive curvature current. Then again by [H, p.71, Theorem 6], we see that $\hat{h}_{can}$ extends to a singular hermitian metric on $K_{X/S}$ with semipositive curvature current on the whole X. This completes the proof of the assertion (1) in Theorem 1.13.

3.4 Completion of the proof of Theorem 1.13

To complete the proof of Theorem 1.13, we need to show that $\hat{h}_{can}$ defines an AZD for K_{X_s} for every $s \in S^\circ$. To show this fact, we modify the construction of $\hat{K}^A_m$ (cf. (3.2)). Here we do not assume $\dim S = 1$.

Let us fix $s \in S^\circ$ and let $h_{0,s}$ be an AZD with minimal singularities of K_{X_s} constructed as (2.5), i.e.,

$$(3.21) \qquad h_{0,s} := \text{the upper envelope of}$$

$$\inf\left\{h \,|\, h \text{ is a singular hermitian metric on } K_{X_s} \text{ such that } \sqrt{-1}\,\Theta_h \geqq 0 \text{ and } h \geqq h_s\right\},$$

where h_s is a fixed C^∞-hermitian metric on K_{X_s}. Let U be a neighborhood of $s \in S^\circ$ in S° which is biholomorphic to the unit open polydisk Δ^k in $\mathbb{C}^k (k := \dim S)$. On $f^{-1}(U)$, we shall identify $K_{X/S}|U$ with $K_X|U$ by tensoring $f^*(dt_1 \wedge \cdots \wedge dt_k)$, where $(t_1, \cdots, t_k)$ denotes the standard coordinate on Δ^k. By the L^2-extension theorem ([O-T, O]) and the argument modeled after [S1], we have the following lemma which asserts that $\Gamma(X_s, \mathcal{O}_{X_s}(A|X_s + mK_{X_s}))$ contains a "large" linear subspace whose elements are extendable on a neighborhood of X_s.

Lemma 3.5 *Every element of* $\Gamma(X_s, \mathcal{O}_{X_s}(A|X_s+mK_{X_s})\otimes\mathcal{I}(h^{m-1}_{0,s}))$ *extends to an element of* $\Gamma(f^{-1}(U), \mathcal{O}_X(A+mK_X))$ *for every positive integer* m. □

Remark 3.6 *In the proof of Lemma 3.5, we only use the pseudoeffectivity of* K_{X_s}. *Hence this lemma implies that all the fiber over* U *has pseudoeffective canonical bundles.* □

Proof of Lemma 3.5. We prove the lemma by induction on m. If $m = 1$, then the L^2-extension theorem ([O-T, O]) implies that every element of $\Gamma(X_s, \mathcal{O}_{X_s}(A + K_{X_s}))$ extends to an element of $\Gamma(f^{-1}(U), \mathcal{O}_X(A + K_X))$. Let $\{\sigma^{(m-1)}_{1,s}, \cdots, \sigma^{(m-1)}_{N(m-1)}\}$ be a basis of $\Gamma(X_s, \mathcal{O}_{X_s}(A|X_s + (m-1)K_{X_s}) \otimes \mathcal{I}(\tilde{h}^{m-2}_{0,s}))$ for some $m \geqq 2$. Suppose that we have already constructed holomorphic extensions:

$$(3.22) \qquad \{\tilde{\sigma}^{(m-1)}_{1,s}, \cdots, \tilde{\sigma}^{(m-1)}_{N(m-1),s}\} \subset \Gamma(f^{-1}(U), \mathcal{O}_X(A + (m-1)K_X))$$

of $\{\sigma^{(m-1)}_{1,s}, \cdots, \sigma^{(m-1)}_{N(m-1),s}\}$ to $f^{-1}(U)$. We define the singular hermitian metric $\tilde{h}_{m-1}$ on $(A + (m-1)K_X)|f^{-1}(U)$ by

$$(3.23) \qquad \tilde{h}_{m-1} := \frac{1}{\sum_{j=1}^{N(m-1)} |\tilde{\sigma}^{(m-1)}_{j,s}|^2}.$$

We note that by the choice of A, $\mathcal{O}_{X_s}(A\,|X_s + mK_{X_s}) \otimes \mathcal{I}(h^{m-1}_{0,s})$ is globally generated. Hence we see that

$$(3.24) \qquad \mathcal{I}(h^m_{0,s}) \subseteq \mathcal{I}(h^{m-1}_{0,s}) \subseteq \mathcal{I}(\tilde{h}_{m-1}|X_s)$$

hold on X_s. Apparently $\tilde{h}_{m-1}$ has a semipositive curvature current. Hence by the L^2-extension theorem ([O-T, p.200, Theorem]), we may extend every element of

$$(3.25) \qquad \Gamma(X_s, \mathcal{O}_{X_s}(A + mK_{X_s}) \otimes \mathcal{I}(h^{m-1}_{0,s}))$$

to an element of

$$(3.26) \qquad \Gamma(f^{-1}(U), \mathcal{O}_X(A + mK_X) \otimes \mathcal{I}(\tilde{h}_{m-1})).$$

This completes the proof of Lemma 3.5 by induction. □

We set

$$(3.27) \quad \Xi^A_{m,s} := \sup\left\{|\,\sigma\,|^{\frac{2}{m}};\, \sigma \in \Gamma(X_s, \mathcal{O}_{X_s}(A|X_s + mK_{X_s}) \otimes \mathcal{I}(h^{m-1}_{0,s})),\ \left|\int_{X_s} h^{\frac{1}{m}}_A \cdot (\sigma \wedge \bar{\sigma})^{\frac{1}{m}}\right| = 1\right\},$$

where sup denotes the pointwise supremum.

Next we shall compare $\Xi^A_{m,s}$ with $\hat{K}^A_{m.s}$. But since $\dim S > 1$, we need to generalize the definition of $\hat{K}^A_{m,s}$. Recall that in the case of $\dim S = 1$, we have defined $\hat{K}^A_{m,s}$ as (3.2). In this case $E_m = f_*\mathcal{O}_X(A + mK_{X/S})$ (cf. (3.1)) may not be locally free on S°. For $s \in S^\circ$ we define $E_{m,s}$ by

$$E_{m,s} := \{\, \sigma \in \Gamma(X_s, \mathcal{O}_{X_s}(A|X_s + mK_{X_s}))|\ \sigma \text{ is extendable to a holomorphic section of } A + mK_{X/S} \text{ on a neighborhood of } X_s\}. \tag{3.28}$$

This is the right substitute of the fiber of E_m at s in this case. For every $s \in S^\circ$, we define $\hat{K}^A_{m,s}$ by

$$\hat{K}^A_{m,s} = \sup\left\{ |\ \sigma\ |^{\frac{2}{m}};\ \sigma \in E_{m,s},\ \left|\int_{X_s} h_A^{\frac{1}{m}} \cdot (\sigma \wedge \bar{\sigma})^{\frac{1}{m}}\right| = 1\right\}. \tag{3.29}$$

This is the extension of the definition (3.2) in Section 3.1, where we have assumed that $\dim S = 1$. And we set

$$\hat{K}^A_{\infty,s} := \limsup_{m\to\infty} h_A^{\frac{1}{m}} \cdot \hat{K}^A_{m,s}. \tag{3.30}$$

On the other hand we have already defined $\hat{h}_{can,A}$ over X (cf. (3.14)). And we set

$$\hat{K}^A_\infty = \hat{h}^{-1}_{can,A}. \tag{3.31}$$

By the definition of $E_{m,s}$ (cf. (3.28)) and the lower-semicontinuity of $\hat{h}_{can,A}$, we have that

$$\hat{K}^A_{\infty,s} \leqq \hat{K}^A_\infty|X_s \tag{3.32}$$

holds for every $s \in S^\circ$. By Lemma 3.5, we obtain the following lemma immediately.

Lemma 3.7

$$\limsup_{m\to\infty} h_A^{\frac{1}{m}} \cdot \Xi^A_{m,s} \leqq \hat{K}^A_\infty|X_s \tag{3.33}$$

holds. □

Proof. By the definition of $\Xi^A_{m,s}$ above and Lemma 3.5 we have that

$$\Xi^A_{m,s} \leqq \hat{K}^A_{m,s} \tag{3.34}$$

holds on X_s. On the other hand, by (3.30) and (3.32), we see that

$$\limsup_{m\to\infty} h_A^{\frac{1}{m}} \cdot \hat{K}^A_{m,s} = \hat{K}^A_{\infty,s} \leqq \hat{K}^A_\infty|X_s \tag{3.35}$$

hold. Hence combining (3.34) and (3.35), we complete the proof of Lemma 3.7. □

We set

$$H_{m,A,s} := (\Xi^A_{m,s})^{-1}. \tag{3.36}$$

Then we have the following lemma.

Lemma 3.8 *If we define*

$$\Xi^A_{\infty,s} := \limsup_{m\to\infty} h_A^{\frac{1}{m}} \cdot \Xi^A_{m,s} \tag{3.37}$$

and

$$H_{\infty,A,s} := \text{the lower envelope of } \Xi^{-1}_{\infty.A,s}, \tag{3.38}$$

$H_{\infty,A,s}$ *is an AZD on* K_{X_s} *with minimal singularities.* □

Proof. Let $h_{0,s}$ be an AZD on K_{X_s} with minimal singularities as (3.21). We note that $\mathcal{O}_{X_s}(A|X_s + mK_{X_s}) \otimes \mathcal{I}(h_{0,s}^{m-1})$ is globally generated by the definition of A. Then by the definition of $\Xi^{A}_{m,s}$,

$$\mathcal{I}(h_{0,s}^m) \subseteq \mathcal{I}(H^m_{m,A,s}) \tag{3.39}$$

holds for every $m \geqq 1$. Hence by repeating the argument in Section 2.2, similar to Lemma 2.5, we have that

$$H_{\infty,A,s} \leqq \left(\int_{X_s} h_{0,s}^{-1}\right) \cdot h_{0,s} \tag{3.40}$$

holds. Hence $H_{\infty,A,s}$ is an AZD on K_{X_s} with minimal singularities. □

By the construction of $\hat{h}_{can}$ and Lemma 3.5

$$\hat{h}_{can}|X_s \leqq H_{\infty,A,s} \tag{3.41}$$

holds on X_s. Hence by Lemma 3.8 and (3.41), we see that $\hat{h}_{can}|X_s$ is an AZD on K_{X_s} with minimal singularities. Since $s \in S^\circ$ is arbitrary, we see that $\hat{h}_{can}|X_s$ is an AZD on K_{X_s} with minimal singularities for every $s \in S^\circ$. This completes the proof of the assertion (2) in Theorem 1.13.

We have already seen that the singular hermitian metric $\hat{h}_{can}$ has semipositive curvature current (cf. Section 3.2). For every $\ell, m \geqq 1$, we set

$$E^{(\ell)}_m = f_*\mathcal{O}_{X_s}(\ell A + mK_{X_s}). \tag{3.42}$$

We note that there exists the union F of at most countable proper subvarieties of S° such that for every $s \in S^\circ \setminus F$ and every $\ell, m \geqq 1$, $E^{(\ell)}_m$ is locally free at s and

$$E^{(\ell)}_{m,s} = \Gamma(X_s, \mathcal{O}_{X_s}(\ell A|X_s + mK_{X_s})) \tag{3.43}$$

holds, where $E^{(\ell)}_{m,s}$ denotes the fiber of the vector bundle $E^{(\ell)}_m$ at s. Then by the construction and Theorem 2.11(see Remark 2.12)[3] for every $s \in S^\circ \setminus F$,

$$\hat{h}_{can}|\, X_s \leqq \hat{h}_{can,s} \tag{3.44}$$

holds, where $\hat{h}_{can,s}$ is the supercanonical AZD on K_{X_s}. This completes the proof of the first half of the assertion (3) in Theorem 1.13. Here the strict inequality may occur on S° by the effect of the fact that we have taken the lower-semicontinuous envelope in the construction of $\hat{h}_{can}$. By the construction it is clear that the latter half of the assertion (3) holds. This completes the proof of Theorem 1.13. □

3.5 Proof of Corollary 1.14

Although I believe that Corollary 1.14 is a immediate consequence of Theorem 1.13, to avoid unnecessary misunderstanding, I give a brief proof here.

Let $f : X \to S$ be a smooth projective family such that K_{X_s} is pseudoeffective for every $s \in S$. We may and do assume that S is the unit open disk Δ in $\mathbb{C}$. We note that there exists a Stein Zariski open subset U of X such that $K_{X/S}|U$ is trivial. Then by the L^2-extension theorem ([O-T, p.200, Theorem]) and the assertion (1) of Theorem 1.13, every element of $H^0(X_s, \mathcal{O}_{X_s}(mK_{X_s}) \otimes \mathcal{I}(\hat{h}^{m-1}_{can}|X_s))$ extends to an element of $H^0(X, \mathcal{O}_X(K_X + (m-1)K_{X/S}) \otimes \mathcal{I}(\hat{h}^{m-1}_{can}))$ for every $s \in S$. By the assertion (2) of Theorem 1.13, we see that

$$H^0(X_s, \mathcal{O}_{X_s}(mK_{X_s}) \otimes \mathcal{I}(\hat{h}^{m-1}_{can}|X_s)) \simeq H^0(X_s, \mathcal{O}_{X_s}(mK_{X_s})) \tag{3.45}$$

holds for every $s \in S$. Hence every element of $H^0(X_s, \mathcal{O}_{X_s}(mK_{X_s}))$ extends to an element of $H^0(X, \mathcal{O}_X(K_X + (m-1)K_{X/S}) \otimes \mathcal{I}(\hat{h}^{m-1}_{can}))$. Then since s is arbitrary, by the upper-semicontinuity of cohomologies, we see that the m-genus $h^0(X_s, \mathcal{O}_{X_s}(mK_{X_s}))$ is locally constant on S. □

[3] Theorem 2.11 is used because some ample line bundle on the fiber may not extends to an ample line bundle on X in general.

3.6 Tensoring semipositive $\mathbb{Q}$-line bundles

In this subsection, we shall consider a minor generalization of Theorems 1.7 and 1.13 and complete the proof of Theorem 1.12.

Let X be a smooth projective n-fold such that the canonical bundle K_X is pseudoeffective. Let A be a sufficiently ample line bundle such that for every pseudoeffective singular hermitian line bundle (L, h_L) on X, $\mathcal{O}_X(A+L)\otimes\mathcal{I}(h_L)$ and $\mathcal{O}_X(K_X+A+L)\otimes\mathcal{I}(h_L)$ are globally generated.

Bet (B, h_B) be a $\mathbb{Q}$-line bundle on X with C^∞-hermitian metric with semipositive curvature. For every $x\in X$ and a positive integer m such that mB is Cartier, we set

$$(3.46)\qquad \hat{K}^A_m(B,h_B)(x) := \sup\left\{ \mid\sigma\mid^{\frac{2}{m}}(x);\ \sigma\in\Gamma(X,\mathcal{O}_X(A+m(K_X+B))), \parallel\sigma\parallel_{\frac{1}{m}} = 1\right\},$$

where

$$(3.47)\qquad \parallel\sigma\parallel_{\frac{1}{m}} := \left|\int_X h_A^{\frac{1}{m}}\cdot h_B\cdot(\sigma\wedge\bar{\sigma})^{\frac{1}{m}}\right|^{\frac{m}{2}}.$$

Then $h_A^{\frac{1}{m}}\cdot\hat{K}^A_m(B,h_B)$ is a continuous semipositive $|B|^2$-valued (n,n)-form on X. Under the above notations, we have the following theorem.

Theorem 3.9 *We set*

$$(3.48)\qquad \hat{K}^A_\infty(B,h_B) := \limsup_{m\to\infty} h_A^{\frac{1}{m}}\cdot\hat{K}^A_m(B,h_B)$$

and

$$(3.49)\qquad \hat{h}_{can,A}(B,h_B) := \text{the lower envelope of } \hat{K}^A_\infty(B,h_B)^{-1}.$$

Then $\hat{h}_{can,A}(B,h_B)$ is an AZD on K_X+B. And we define

$$(3.50)\qquad \hat{h}_{can}(B,h_B) := \text{the lower envelope of } \inf_A \hat{h}_{can,A}(B,h_B),$$

where inf *denotes the pointwise infimum and A runs all the ample line bundles on X. Then $\hat{h}_{can}(B,h_B)$ is a well defined AZD on K_X+B with minimal singularities (cf. Definition 5.2) depending only on X and (B,h_B).* □

The proof of Theorem 3.9 is parallel to that of Theorem 1.7. Hence we omit it. We also have the following generalization of Theorem 1.13.

Theorem 3.10 *Let $f: X\longrightarrow S$ be a proper surjective projective morphism with connected fibers between complex manifolds such that for a general fiber X_s, K_{X_s} is pseudoeffective. We set S° be the maximal nonempty Zariski open subset of S such that f is smooth over S° and $X^\circ = f^{-1}(S^\circ)$. Let (B,h_B) be a $\mathbb{Q}$-line bundle on X with C^∞-hermitian metric h_B with semipositive curvature on X. Then there exists a unique singular hermitian metric $\hat{h}_{can}(B,h_B)$ on $K_{X/S}+B$ depending only on h_B such that*

(1) $\hat{h}_{can}(B,h_B)$ has semipositive curvature,

(2) $\hat{h}_{can}(B,h_B)|X_s$ is an AZD on $K_{X_s}+B|X_s$ with minimal singularities for every $s\in S^\circ$,

(3) There exists the union F' of at most countable union of proper subvarieties of S° such that for every $s\in S^\circ\setminus F'$,

$$(3.51)\qquad \hat{h}_{can}(B,h_B)\,|\,X_s \leqq \hat{h}_{can}((B,h_B)|X_s)$$

holds. And $\hat{h}_{can}(B,h_B)|X_s = \hat{h}_{can}((B,h_B)|X_s)$ holds outside of a set of measure 0 on X_s for almost every $s\in S^\circ$. □

Proof. The proof is almost parallel to that of Theorem 1.13. Let q be the minimal positive integer such that qB is a genuine line bundle. The only difference in the proof is that we extend

$$H^0(X_s, \mathcal{O}_{X_s}(A|X_s + (mK_X + q\lfloor m/q \rfloor B_s)) \otimes \mathcal{I}(\hat{h}_{can}(B, h_B)|X_s)^{m-1}))$$

by the induction on m similarly as in Lemma 3.5, where A is a sufficiently ample line bundle on X independent of m. The rest of the proof is completly the same. Hence we omit it. $\square$

By Theorem 3.10 and the L^2-extension theorem ([O-T, p.200, Theorem]), we obtain the following corollary immediately.

Corollary 3.11 *([S2]) Let $f : X \longrightarrow S$ be a smooth projective family over a complex manifold S and let (B, h_B) be a $\mathbb{Q}$-line bundle with C^∞-hermitian metric h_B with semipositive curvature on X. Then for every $m \geqq 1$ such that mB is Cartier, the twisted m-genus $h^0(X_s, \mathcal{O}_{X_s}(m(K_{X_s} + B|X_s)))$ is a locally constant function on S.* $\square$

Proof. We may and do assume that S is the unit open disk Δ in $\mathbb{C}$. By the L^2-extension theorem ([O-T, p.200, Theorem]) and the assertion (1) of Theorem 3.10, for every $s \in S$, every element of

$$H^0(X_s, \mathcal{O}_X(m(K_{X_s} + B|X_s) \otimes \mathcal{I}((\hat{h}_{can}(B, h_B)^{m-1}|X_s) \cdot h_B))$$

extends to an element of

$$H^0(X, \mathcal{O}_X(K_X + B + (m-1)(K_{X/S} + B)) \otimes \mathcal{I}(\hat{h}_{can}(B, h_B)^{m-1} \cdot h_B)).$$

By the assertion (2) of Theorem 3.10, we see that for every $s \in S$

$$H^0(X_s, \mathcal{O}_X(m(K_{X_s} + B|X_s) \otimes \mathcal{I}((\hat{h}_{can}(B, h_B)^{m-1}|X_s) \cdot h_B)) \simeq H^0(X_s, \mathcal{O}_X(m(K_{X_s} + B|X_s)) \tag{3.52}$$

holds. Hence every element of $H^0(X_s, \mathcal{O}_{X_s}(m(K_{X_s} + B|X_s))$ extends to an element of $H^0(X, \mathcal{O}_X(K_X + B + (m-1)(K_{X/S} + B)) \otimes \mathcal{I}(\hat{h}_{can}(B, h_B)^{m-1}))$.

Since s is arbitrary, by the upper-semicontinuity of cohomologies, we see that the twisted m-genus $h^0(X_s, \mathcal{O}_{X_s}(m(K_{X_s} + B|X_s)))$ is locally constant on S. $\square$

The following corollary slightly improves Theorems 1.13 and 3.10.

Corollary 3.12 *The sets F and F' in Theorems 1.13 and 3.10 respectively do not exist.* $\square$

Proof. We shall prove that F is empty. We note that $E_m = f_*\mathcal{O}_X(A + mK_{X/S})$ (cf. (3.1)) used to define $\hat{h}_{can}$ is locally free over S°, since $h^0(X_s, \mathcal{O}_X(A|X_s + mK_{X_s}))(s \in S^\circ)$ is locally constant over S° by Corollary 3.11 and every element of $H^0(X_s, \mathcal{O}_X(A|X_s + mK_{X_s}))(s \in S^\circ)$ extends to a holomorphic section of $\mathcal{O}_X(A + mK_{X/S})$. By the same reason, for every $\ell, m \geqq 1$, we see that $E_m^{(\ell)} = f_*\mathcal{O}_X(\ell A + mK_{X/S})$ is locally free over S° and every element of $H^0(X_s, \mathcal{O}_X(\ell A|X_s + mK_{X_s}))(s \in S^\circ)$ extends to a holomorphic section of $\mathcal{O}_X(\ell A + mK_{X/S})$. Then by the construction of $\hat{h}_{can}$ in Section 3.1, viewing the last part of Section 3.4 (see the definition of F just before (3.43)), we see that F ought to be empty. The emptyness of F' follows from the parallel argument. $\square$

Now we complete the proof of Theorem 1.12. $\square$

4 Generalization to KLT pairs

In this section we shall generalize Theorems 1.7 and 1.12 to the case of KLT pairs. This leads us to the proof of the invariance of logarithmic plurigenera (Theorem 1.15). Here the essential new techniques are the perturbation of the log canonical bundle by ample $\mathbb{Q}$-line bundles (cf. Section 4.5) and the use of the dynamical systems of singular hermitian metrics (cf. Section 4.6). Here we make use the flexibility of $\mathbb{Q}$-line bundles and the nice convergence properties of $L^{2/m}$-norms.

4.1 Statement of the fundamental results

First we shall recall the notion of KLT pairs.

Definition 4.1 *Let X be a normal variety and let $D = \sum_i d_i D_i$ be an effective* **Q***-divisor such that $K_X + D$ is* **Q***-Cartier. If $\mu : Y \longrightarrow X$ is a log resolution of the pair (X, D), i.e., μ is a composition of successive blowing ups with smooth centers such that Y is smooth and the support of $\mu^* D$ is a divisor with simple normal crossings, then we can write*

$$K_Y + \mu_*^{-1} D = \mu^*(K_X + D) + F$$

with $F = \sum_j e_j E_j$ for the exceptional divisors $\{E_j\}$, where $\mu_*^{-1} D$ denotes the strict transform of D. We call F the discrepancy and $e_j \in \mathbf{Q}$ the discrepancy coefficient for E_j. We define the **log discrepancy***: $\mathrm{ld}(E_j; X, D)$ at E_j by $\mathrm{ld}(E_j; X, D) := e_j + 1$.*

The pair (X, D) is said to be **KLT***(Kawamata log terminal) (resp.* **LC***(log canonical)), if $d_i < 1$(resp. $\leqq 1$) for all i and $e_j > -1$ (resp. $\geqq -1$) for all j for a log resolution $\mu : Y \longrightarrow X$. For a pair (X, D) with D effective, we define the multiplier ideal sheaf $\mathcal{I}(D)$ of (X, D) by $\mathcal{I}(D) := \mu_* \mathcal{O}(\lceil F \rceil)$ and $CLC(X, D) = \mathrm{Supp}\, \mathcal{O}_X / \mathcal{I}(D)$. We call $CLC(X, D)$* **the center of log canonical singularities** *of (X, D). In this terminnology (X, D) (with D effective) is KLT, if and only if $CLC(X, D) = \emptyset$.*

For an irreducible closed subset W in X, we set

$$\mathrm{mld}(\mu_W; X, D) := \inf_{c_X(E) = \mu_W} \mathrm{ld}(E; X, D)$$

and call it the **minimal log discrepancy** *at the generic point of W with respect to (X, D), where μ_W denotes the generic point of W and the infimum is taken over the all effective Cartier divisors E on models of X whose ceneter $c_X(E)$ is equal to W.* □

The following is the counterpart of Theorem 1.7 in the KLT case.

Theorem 4.2 *Let (X, D) be a KLT pair such that X is a smooth projective variety. Suppose that $K_X + D$ is pseudoeffective.*

Then there exists a singular hermtian metric $\hat{h}_{can}$ on $K_X + D$ such that

(1) $\hat{h}_{can}$ is uniquely determined by the pair (X, D) (see Remark 4.5 below for the precise meaning of the uniqueness),

(2) $\hat{h}_{can}$ is an AZD on $K_X + D$, i.e.,

(a) $\sqrt{-1}\,\Theta_{\hat{h}_{can}}$ is a closed semipositive current,

(b) $H^0(X, \mathcal{O}_X(m(K_X + D)) \otimes \mathcal{I}(\hat{h}_{can}^m)) \simeq H^0(X, \mathcal{O}_X(m(K_X + D)))$ holds for every $m \geqq 1$ such that mD is an integral divisor [4].

Moreover $\hat{h}_{can}$ is an AZD with minimal singularities (cf. Definition 5.2).

We call $\hat{h}_{can}$ in Theorem 4.2 the supercanonical AZD on $K_X + D$ on (X, D). □

In Theorem 4.2 we have used the same notation $\hat{h}_{can}$ as in Theorem 1.10 for simplicity. I think this will cause no confusion. We call $\hat{h}_{can}$ in Theorem 4.2 constructed of $K_X + D$ as in Theorem 1.7. The construction is essentially parallel to Theorem 1.7 and will be given in the next subsection (cf. Theorem 4.4)

As before, we study the variation of the supercanonical AZD's for KLT pairs and prove the following semipositivity theorem similar to the non logarithmic case in Theorem 1.7.

Theorem 4.3 *Let $f : X \longrightarrow S$ be a proper surjective projective morphism between complex manifolds with connected fibers and let D be an effective $\mathbb{Q}$-divisor on X such that*

[4] Without this condition $\mathcal{I}(\hat{h}_{can}^m)$ is not well defined.

(a) *D is $\mathbb{Q}$-linearly equivalent to a $\mathbb{Q}$-line bundle B,*

(b) *The set: $S^\circ := \{s \in S |\ f$ is smooth over s and (X_s, D_s) is KLT $\}$ is nonempty,*

(c) *For every $X_s (s \in S^\circ)$, $K_{X_s} + D_s$ is pseudoeffective* [5].

Then there exists a singular hermitian metric $\hat{h}_{can}$ on $K_{X/S} + B$ such that

(1) *$\hat{h}_{can}$ has semipositive curvature current,*

(2) *$\hat{h}_{can} | X_s$ is an AZD on $K_{X_s} + B_s$ for every $s \in S^\circ$,*

(3) *For every $s \in S^\circ$, $\hat{h}_{can} \mid X_s \leqq \hat{h}_{can,s}$ holds, where $\hat{h}_{can,s}$ denotes the supercanonical AZD on $K_{X_s} + B_s$. And $\hat{h}_{can}|X_s = \hat{h}_{can,s}$ holds outside of a set of measure 0 on X_s for almost every $s \in S^\circ$.*

□

In the proof of Theorem 4.3, we do need to use the fact that D is effective, since we need to use the semipositivity result similar to Lemma 3.1.

4.2 Construction of the supercanonical AZD's for KLT pairs

In this subsection, we shall construct the supercanonical AZD's for KLT pairs similarly to Theorem 1.7. Let (X, D) be a sub KLT pair such that X is smooth and $K_X + D$ is pseudoeffective. In this subsection we shall consider D as a $\mathbb{Q}$-line bundle. Because we are considering singular hermitian metrics on $K_X + D$ or its multiples, this is not a problem.

Let A be a sufficiently ample line bundle such that for any pseudoeffective singular hermitian line bundle (L, h_L) (cf. Definition 1.6), $\mathcal{O}_X(A + L) \otimes \mathcal{I}(h_L)$ and $\mathcal{O}_X(A + K_X + L) \otimes \mathcal{I}(h_L)$ are globally generated. Such an ample line bundle A exists by Proposition 5.1 below. Let $D = \sum_i d_i D_i$ be the irreducible decomposition of D and for every i we choose a nonzero global holomorphic section σ_{D_i} of $\mathcal{O}_X(D_i)$ with divisor D_i. For every positive integer m such that $mD \in \mathrm{Div}(X)$, we set

$$\hat{K}^A_m := \sup\left\{ |\sigma|^{\frac{2}{m}}; \sigma \in \Gamma(X, \mathcal{O}_X(A + m(K_X + D))), \| \sigma \|_{\frac{1}{m}} = 1 \right\}, \tag{4.1}$$

where sup denotes the pointwise supremum and

$$\| \sigma \|_{\frac{1}{m}} := \left| \int_X h_A^{\frac{1}{m}} \cdot h_D \cdot (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right|,$$

where

$$h_D := \frac{1}{\prod_i |\sigma_{D_i}|^{2d_i}}. \tag{4.2}$$

Here $\mid \sigma \mid^{\frac{2}{m}}$ is not a function on X, but the supremum is takan as a section of the real line bundle $|A|^{\frac{2}{m}} \otimes |K_X + D|^2$ in the obvious manner. Then

$$\hat{h}_{m,A} := (\hat{K}^A_m)^{-1} \tag{4.3}$$

is a singular hermitian metric on $m^{-1}A + (K_X + D)$ with semipositive curvature current. Then $h_A^{-1/m} \cdot \hat{h}_{m,A}$ is a singular hermitian metric on $K_X + D$. Then

$$\hat{h}_{can,A} := \text{the lower envelope of } \liminf_{m\to\infty} h_A^{-\frac{1}{m}} \cdot \hat{h}_{m,A} \tag{4.4}$$

is a singular hermitian metric on $K_X + D$ with semipositive curvature current,where m runs all the integers such that $mD \in \mathrm{Div}(X)$. Now we have the following theorem similar to Theorem 1.7.

[5] Here actually we only need to assume that for some fiber $X_s (s \in S^\circ)$. (X_s, D_s) is KLT and $K_{X_s} + D_s$ is pseudoeffective. See Theorem 4.10 below.

Lemma 4.4 *$\hat{h}_{can,A}$ and*

$$\hat{h}_{can} := \text{the lower envelope of } \inf_A h_{can,A} \tag{4.5}$$

are well defined and are AZD's of $K_X + D$ with minimal singularities, where A runs all the sufficiently ample line bundles on X. $\square$

Proof. The proof of Lemma 4.4 is almost parallel to the proof of Theorem 1.10. Let h_0 be an AZD with minimal singularities on $K_X + D$ constructed in ([D-P-S, Theorem 1.5]) as in Section 5.2. Then by Hölder's inequality,

$$\left| \int_X h_A^{\frac{1}{m}} \cdot h_D \cdot (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right| \leqq \left(\int_X h_A \cdot h_0^m \cdot \mid \sigma \mid^2 \cdot (h_0^{-1} \cdot h_D) \right)^{\frac{1}{m}} \cdot \left(\int_X h_0^{-1} \cdot h_D \right)^{\frac{m-1}{m}},$$

holds, where m is a positive integer m such that $mD \in \mathrm{Div}(X)$, and $\sigma \in \Gamma(X, \mathcal{O}_X(A + m(K_X + D))$. This inequality makes sense, because

$$\int_X h_0^{-1} \cdot h_D < +\infty$$

holds, since (X, D) is KLT. Hence we have the inequality:

$$\hat{K}_m^A \geqq K(A + m(K_X + D), h_A \cdot h_0^{m-1} \cdot h_D)^{\frac{1}{m}} \cdot \left(\int_X h_0^{-1} \cdot h_D \right)^{-\frac{m-1}{m}}.$$

And by Lemma 2.3 and Remark 2.4, if A is sufficiently ample, letting m tend to infinity, we have that

$$\hat{h}_{can,A} \leqq h_0 \cdot \left(\int_X h_0^{-1} \cdot h_D \right). \tag{4.6}$$

holds. On the other hand the upper estimate of $\hat{K}_m^A$ is obtained by the submeanvalue inequality for plurisubharmonic functions.

Let $\sigma \in \Gamma(X, \mathcal{O}_X(A + m(K_X + D)))$ for some m such that mD is integral. Let $(U, (z_1, \cdots, z_n))$ be a local coordinate on X which is biholomorphic to the unit open polydisk Δ^n by the coordinate $(z_1, \cdots, z_n)$. Taking U sufficiently small, we may assume that $(z_1, \cdots, z_n)$ is a holomorphic local coordinate on a neighborhood of the closure of U and there exist local holomorphic frames $\mathbf{e}_A$ of A and $\mathbf{e}_{D_i}$ of D_i for every i respectively on a neighborhood of the closure of U. Then there exists a bounded holomorphic function f_U on U such that

$$\sigma = f_U \cdot \mathbf{e}_A \cdot (dz_1 \wedge \cdots \wedge dz_n)^m \cdot \left(\prod_i \mathbf{e}_{D_i}^{d_i} \right)^m \tag{4.7}$$

holds. Suppose that

$$\left| \int_X h_A^{\frac{1}{m}} \cdot h_D \cdot (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right| = 1 \tag{4.8}$$

holds. Then since $h_D \cdot \prod_i |\mathbf{e}_{D_i}|^{2d_i}$ and $h_A(\mathbf{e}_A, \mathbf{e}_A)$ has positive lower bound on U, as (2.3) by the submeanvalue inequality for plurisubharmonic functions, we see that $|f_U|^{\frac{2}{m}}$ is bounded compact uniformly on U. Hence just as Lemma 2.1, we have that there exists a positive constant C such that

$$\limsup_{m \to \infty} h_A^{\frac{1}{m}} \cdot h_D \cdot \hat{K}_m^A \leqq C \cdot \frac{dV}{\prod_i \parallel \sigma_{D_i} \parallel^{2d_i}} \tag{4.9}$$

holds, where for every i, $\parallel \sigma_{D_i} \parallel$ denotes the hermitian norm of σ_{D_i} with respect to a fixed C^∞-hermitian metric on D_i and dV is a fixed C^∞-volume form on X.

Combining (4.6) and (4.9), $\hat{h}_{can,A}$ is not identically 0 and is an AZD on $K_X + D$ with minimal singularities. This completes the proof of Lemma 4.4. $\square$

By Lemma 4.4 and the definiton of $\hat{h}_{can}$, $\hat{h}_{can}$ is an AZD on $K_X + D$ with minimal singularities indeed. And the rest of the proof is similar to that of Theorem 1.7. This completes the proof of Theorem 4.2. $\square$

Remark 4.5 *In the above proof of Theorem 4.2, $\hat{K}^A_m$ depends on the choice of $\{\sigma_{D_i}\}$. Nevertheless the singular volume form:*

$$\frac{h_A^{\frac{1}{m}} \cdot \hat{K}^A_m}{\prod_i |\sigma_{D_i}|^{2d_i}}$$

does not depend on the choice. Hence we see that the singular volume form:

$$\frac{\hat{h}^{-1}_{can}}{\prod_i |\sigma_{D_i}|^{2d_i}}$$

is uniquely determined by (X, D). In other words, $\hat{h}^{-1}_{can}$ is uniquely determined as a singular volume form on X and it does not depend on the choice of $\{\sigma_{D_i}\}$. $\square$

4.3 Construction of supercanonical AZD's on adjoint line bundles

Let (L, h_L) be a KLT singular hermitian $\mathbb{Q}$-line bundle (cf. Definition 1.18) on a smooth projective variety X. Suppose that $K_X + L$ is pseudoeffective. Let A be a sufficiently ample line bundle on X in the sense of Proposition 5.1 in Appendix and let h_A be a C^∞-hermitian metric on A. For a positive integer m such that mL is a genuine line bundle and $\sigma \in \Gamma(X, \mathcal{O}_X(A + m(K_X + L)))$, we set

$$\| \sigma \|_{\frac{1}{m}} := \left| \int_X h_A^{\frac{1}{m}} \cdot h_L \cdot (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right|^{\frac{m}{2}}. \tag{4.10}$$

For $x \in X$, we set

$$\hat{K}^A_m(x) := \sup \left\{ | \sigma |^{\frac{2}{m}} (x) \mid \sigma \in \Gamma(X, \mathcal{O}_X(A + m(K_X + L))), \| \sigma \|_{\frac{1}{m}} = 1 \right\}. \tag{4.11}$$

We note that $\| \sigma \|_{\frac{1}{m}}$ is well defined by the assumption that (L, h_L) is KLT. We set

$$\hat{h}_{can,A}(L, h_L) := \text{the lower envelope of } \liminf_{m \to \infty} h_A^{-\frac{1}{m}} \cdot (\hat{K}^A_m)^{-1} \tag{4.12}$$

and

$$\hat{h}_{can}(L, h_L) := \text{the lower envelope of } \inf_A \hat{h}_{can,A}(L, h_L), \tag{4.13}$$

where A runs all the ample line bundles on X.

Theorem 4.6 *$\hat{h}_{can,A}(L, h_L)$ and $\hat{h}_{can}(L, h_L)$ defined respectively as (4.12) and (4.13) are AZD's of $K_X + L$ with minimal singularities. We call $\hat{h}_{can}(L, h_L)$ the supercanonical AZD on $K_X + L$ with respect to h_L.* $\square$

The proof of Theorem 4.6 is completely parallel to the one of Theorem 4.2 above. In fact, for example the lower estimate of $\hat{K}^A_m$ as follows. Let h_0 is a AZD with minimal singularities on $K_X + L$. Then similarly as 2.13) the inequality:

$$\left| \int_X h_A^{\frac{1}{m}} \cdot h_L \cdot (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right| \leqq \left(\int_X h_A \cdot h_0^m \cdot | \sigma |^2 \cdot (h_0^{-1} \cdot h_L) \right)^{\frac{1}{m}} \cdot \left(\int_X h_0^{-1} \cdot h_L \right)^{\frac{m-1}{m}},$$

holds. where m is a positive integer m such that mL is a genuine line bundle and $\sigma \in \Gamma(X, \mathcal{O}_X(A + m(K_X + L)))$. We note that this inequality makes sense, since (L, h_L) is KLT. And the rest is similar to that of Theorem 4.2. Hence we omit it.

4.4 Proof of Theorem 4.3; Log general type case

In this subsection, we shall prove Theorem 4.3 under the following additional assumptions:

(1) Every fiber (X_s, D_s) over $s \in S^\circ$ is of log general type, i.e., $K_{X_s} + D_s$ is big,

(2) D is $\mathbb{Q}$-linearly equivalent to a genuine line bundle B.

In the following proof we shall consider D as a line bundle $\mathcal{O}_X(B)$ and we shall abuse the notation $\mathcal{O}_X(D)$ instead of $\mathcal{O}_X(B)$, i.e., we shall fix a line bundle structure associated with the $\mathbb{Q}$-divisor D.

Let $f : X \to S$ and D be as in Theorem 4.3. The construction of $\hat{h}_{can}$ in Theorem 4.3 is similar to that in Theorem 1.13. Here we shall assume that S is of dimension 1 for simplicity. The case of $\dim S > 1$ is treated parallel to Section 3.3. The construction of $\hat{h}_{can}$ on the family is similar to Theorem 1.13. More precisely we replace E_m in (3.1) by

$$E_m := f_* \mathcal{O}_X(A + m(K_{X/S} + D)), \tag{4.14}$$

where A is a sufficiently ample line bundle on X. We note that E_m is locally free by the assumption: $\dim S = 1$. Let $E_{m,s}$ denotes the fiber of the vector bundle E_m at s. Let h_A be a C^∞-hermitian metric on A with strictly positive curvature on X. Let σ_D is a nonzero multivalued holomorphic section of $\mathcal{O}_X(D)$ with divisor D and we set

$$h_D = \frac{1}{|\sigma_D|^2}. \tag{4.15}$$

By fixing σ_D we may identify a holomorphic section τ of $\mathcal{O}_X(m(K_X + D))$ with a (multivalued) meromorphic m-ple canonical form $\tau/(\sigma_D)^m$. For $s \in S^\circ$ we set

$$\hat{K}^A_{m,s} := \sup\{\, |\sigma|^{\frac{2}{m}} ; \sigma \in E_{m,s}, \| \sigma \|_{\frac{1}{m}} = 1\}, \tag{4.16}$$

where

$$\| \sigma \|_{\frac{1}{m}} = \left| \int_{X_s} h_A^{\frac{1}{m}} \cdot h_{D,s} \cdot (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right|^{\frac{m}{2}}, \tag{4.17}$$

where $h_{D,s} := h_D | X_s$. We set

$$\hat{K}^A_{\infty,s} := \text{the upper envelope of } \limsup_{m\to\infty} h_A^{\frac{1}{m}} \cdot \hat{K}^A_{m,s}. \tag{4.18}$$

and define $\hat{K}^A_\infty$ by $\hat{K}^A_\infty | X_s = \hat{K}^A_{\infty,s}$. Then we set

$$\hat{h}_{can,A} := \text{the lower envelope of } (\hat{K}^A_\infty)^{-1}. \tag{4.19}$$

and

$$\hat{h}_{can} := \text{the lower envelope of } \inf_A \hat{h}_{can,A}, \tag{4.20}$$

where A runs all the ample line bundles on X. Since h_D defined as (4.15) has semipositive curvature current, using Theorem 3.1, we see that $\hat{h}_{can}$ has semipositive curvature current on $X^\circ := f^{-1}(S^\circ)$ by the same argument as in the proof of Theorem 1.13 in Section 3.2. And we can extend $\hat{h}_{can}$ to a singular hermitian metric on $K_{X/S} + D$ over the whole X just as in Section 3.2. Hence we only need to prove the assertions (2) and (3) in Theorem 4.3.

For every $s \in S^\circ$, we also define the canonical singular hermitian metric $\hat{h}_{can,s}$ on $K_{X_s} + D_s$ as in Lemma 4.4. We note that $\hat{h}_{can} | X_s$ may be different from $\hat{h}_{can,s}$ for some $s \in S^\circ$. To prove Theorem 4.3, we need to compare $\hat{h}_{can} | X_s$ with $\hat{h}_{can,s}$.

Let us fix $s \in S^\circ$. Let U be a neighborhood of $s \in S^\circ$ (in S°) which is biholomorphic to the unit open

disk Δ in $\mathbb{C}$ by a local coordinate s. Hereafter over $f^{-1}(U)$, we identify $K_{X/S}|U$ with $K_X|U$ by the isomorphism:

$$\otimes\ f^{*}ds : K_{X/S}|U \to K_X|U. \tag{4.21}$$

First we shall assume that $K_{X_s} + D_s$ is big, i.e., (X_s, D_s) is of log general type. The general case follows from this special case by considering $K_{X/S} + D + \epsilon A$ instead of $K_{X/S} + D$ and letting ϵ tend to 0. We shall discuss in detail later.

By [B-C-H-M] there exists a modification

$$\mu_s : Y_s \to X_s$$

such that $\mu_s^*(K_{X_s} + D_s)$ has a Zariski decomposition:

$$\mu_s^*(K_{X_s} + D_s) = P_s + N_s, \tag{4.22}$$

i.e., $P_s, N_s \in \mathrm{Div}(Y_s) \otimes \mathbb{Q}$ such that

(1) P_s is nef,

(2) N_s is effective,

(3) $H^0(X_s, \mathcal{O}_{X_s}(\lfloor m(K_{X_s} + D_s)\rfloor)) \simeq H^0(Y_s, \mathcal{O}_{Y_s}(\lfloor mP_s\rfloor))$ for every $m \geqq 0$.

We note that in this case P_s is semiample (see [B-C-H-M]).

Lemma 4.7 *Let h_{P_s} be a C^∞-hermtian metric on P_s with semipositive curvature and let τ_N be a multivalued holomorphic section of N_s with divisor N_s. Then*

$$h_{P_s} \cdot \frac{1}{|\tau_N|^2}$$

is an AZD on $\mu_s^*(K_{X_s} + D_s)$ with minimal singularities. $\square$

Proof. By Kodaira's lemma $H^0(Y_s, \mathcal{O}_{Y_s}(m_0P_s - \mu_s^*A)) \neq 0$ holds for a sufficiently large positive integer m_0 with m_0P_s is Cartier. We take a nonzero element $\sigma_0 \in H^0(X_s, \mu_{s,*}\mathcal{O}_{Y_s}(m_0P_s - \mu_s^*A))$ and identify σ_0 as an element of $H^0(X_s, \mathcal{O}_{X_s}(m(K_{X_s} + D_s) - A))$ in the natural way. Hence we have the inclusion:

$$\otimes\, \mu_s^*\sigma_0\ : \mathcal{O}_Y(\mu_s^*(A + m(K_s + D_s))) \hookrightarrow \mathcal{O}_Y(\mu_s^*((m + m_0)(K_{X_s} + D_s))).$$

Then for every element $\sigma \in H^0(X_s, \mathcal{O}_{X_s}(A + m(K_{X_s} + D_s)))$, by Hölder's inequality we see that

$$\int_{X_s} |\sigma_0 \cdot \sigma|^{\frac{2}{m+m_0}} \cdot h_D \leqq \left(\int_{X_s} h_A^{\frac{1}{m}} \cdot |\sigma|^{\frac{2}{m}}\right)^{\frac{m}{m+m_0}} \cdot \left(\int_{X_s} h_A^{-\frac{1}{m_0}} \cdot |\sigma_0|^{\frac{2}{m_0}}\right)^{\frac{m_0}{m+m_0}} \tag{4.23}$$

holds. Now for every positive integer ℓ, we set

$$K_\ell := \sup\left\{ \mid \sigma \mid^{\frac{2}{\ell}} ; \sigma \in \Gamma(X_s, \mathcal{O}_X(\ell(K_{X_s} + D_s))), \left|\int_{X_s} h_{D,s} \cdot (\sigma \wedge \bar{\sigma})^{\frac{1}{\ell}}\right| = 1\right\}. \tag{4.24}$$

Then as Theorem 1.3,

$$h_{can} := \text{the lower envelope of } (\limsup_{\ell\to\infty} K_\ell)^{-1} \tag{4.25}$$

is an AZD on $K_{X_s} + D_s$. Then by (4.23) we see that

$$K_{m+m_0} \geqq |\sigma_0|^{\frac{2}{m+m_0}} \cdot \left(\hat{K}_{m,A}\right)^{\frac{m}{m+m_0}} \cdot \left(\int_{X_s} h_A^{-\frac{1}{m_0}} \cdot |\sigma_0|^{\frac{2}{m_0}}\right)^{-\frac{m_0}{m+m_0}}$$

holds. Letting m tend to infinity, we see that

$$(4.26) \qquad h_{can} \leqq \hat{h}_{can,A}$$

holds. By the definition of the Zariski decomposition and the semiampleness of P_s, we see that $h_{P_s} \cdot \frac{1}{|\tau_N|^2}$ is quasi-isometric to h_{can},i.e., the ratio of these metrics is pinched by positive constants. Hence by (4.26) $h_{P_s} \cdot \frac{1}{|\tau_N|^2}$ is an AZD with minimal singularities on $\mu_s^*(K_{X_s} + D_s)$. $\square$

Let a be a positive integer such that $aP_s \in \mathrm{Div}(Y_s)$. Let us take the ample line bundle A so that for every pseudoeffective singular hermitian line bundle (L, h_L) on X_s

$$\mathcal{O}_{X_s}(A|X_s + j(K_{X_s} + D_s) + L) \otimes \mathcal{I}(h_L)$$

is globally generated over X_s for every $0 \leqq j \leqq a$. This is certainly possible (see Proposition 5.1 in Section 5.1).

Let us fix a C^∞-hermitian metric $h_{ref,s}$ on $K_{X_s} + D_s$. The following lemma is similar to Lemma 3.5. Hereafter we shall denote $A|X_s$ (resp. $h_A|X_s$) by A_s (resp. $h_{A,s}$) for simplicity.

Lemma 4.8 *Let h_D and the positive integer a be as above. If we take A sufficiently ample as above, we have the followings.*

(1) For every positive integer m, every element of $\Gamma(X_s, \mathcal{O}_{X_s}(A_s + m(K_{X_s} + D_s)) \otimes \mathcal{I}(h_{D,s} \cdot \hat{h}_{can,s}^{a\lfloor m-1/a \rfloor}))$ extends to an element of $\Gamma\left(f^{-1}(U), \mathcal{O}_X(A + m(K_X + D))\right)$, where $h_{D,s}$ denotes the restriction $h_D|X_s$.

(2) There exists a positive constant C such that

$$\hat{h}_{can}|X_s \leqq C \cdot \hat{h}_{can,s}$$

holds on X_s. In particular $\hat{h}_{can}|X_s$ is an AZD on $K_{X_s} + D_s$ with minimal singularities (cf. Definition 5.2). $\square$

Proof of Lemma 4.8. We prove the lemma by induction on m. If $m = 1$, then the L^2-extension theorem ([O-T, O]) implies that every element of $\Gamma(X_s, \mathcal{O}_{X_s}(A_s + (K_{X_s} + D_s)) \otimes \mathcal{I}(h_{D,s})) = \Gamma(X_s, \mathcal{O}_{X_s}(A + (K_{X_s} + D_s)))$ (because (X_s, D_s) is KLT) extends to an element of $\Gamma(f^{-1}(U), \mathcal{O}_X(A + (K_X + D))$.

Suppose that the extension is settled for $m - 1 (m \geqq 2)$. Let $\{\sigma_{1,s}^{(m-1)}, \cdots, \sigma_{N(m-1)}^{(m-1)}\}$ be a basis of $\Gamma(X_s, \mathcal{O}_{X_s}(A_s + (m-1)(K_{X_s} + D_s) \otimes \mathcal{I}(h_D \cdot \hat{h}_{can}^{a\lfloor (m-2)/a \rfloor}))$. By the inductive assumption, we have already constructed holomorphic extensions:

$$\{\tilde{\sigma}_{1,s}^{(m-1)}, \cdots, \tilde{\sigma}_{N(m-1),s}^{(m-1)}\} \subset \Gamma(f^{-1}(U), \mathcal{O}_X(A + (m-1)(K_X + D)))$$

of $\{\sigma_{1,s}^{(m-1)}, \cdots, \sigma_{N(m-1),s}^{(m-1)}\}$ to $f^{-1}(U)$. We define the singular hermitian metric $\tilde{h}_{m-1}$ on $A + (m - 1)(K_X + D))|f^{-1}(U)$ by

$$(4.27) \qquad \tilde{h}_{m-1} := \frac{1}{\sum_{j=1}^{N(m-1)} |\tilde{\sigma}_{j,s}^{(m-1)}|^2}.$$

By the choice of a and the fact that (X_s, D_s) is KLT, aP_s is integral on Y_s and we have the inclusion:

$$(4.28) \qquad \mathcal{O}_{X_s}\left(A_s + \left(m - 1 - a\left\lfloor \frac{m-2}{a} \right\rfloor\right)(K_{X_s} + D_s)\right) \otimes (\mu_s)_* \mathcal{O}_{Y_s}\left(a\left\lfloor \frac{m-2}{a} \right\rfloor \cdot P_s\right)$$

$$\hookrightarrow \mathcal{O}_{X_s}(A + (m-1)(K_{X_s} + D_s)) \otimes \mathcal{I}(h_{D,s} \cdot \hat{h}_{can,s}^{a\lfloor m-2/a \rfloor}).$$

In fact by Lemma 4.7, $(\mu_s)_* \mathcal{O}_{Y_s}\left(a\left\lfloor \frac{m-2}{a} \right\rfloor \cdot P_s\right)$ is nothing but the sheaf of germs of locally bounded holomorphic sections of $A + a\lfloor (m-2)/a \rfloor \cdot (K_{X_s} + D_s)$ with respect to the metric $\hat{h}_{can,s}^{a\lfloor (m-2)/a \rfloor}$. Then

since (X_s, D_s) is KLT and $\mathcal{O}_{X_s}(A_s + (m-1)(K_{X_s} + D_s)) \otimes \mathcal{I}(h_{D,s} \cdot \hat{h}_{can,s}^{a\lfloor(m-2)/a\rfloor})$ is globally generated over X_s (by the choice of A), (4.28) implies that

$$\tilde{h}_{m-1}|X_s = O(h_{A,s} \cdot \hat{h}_{can,s}^{a\lfloor(m-2)/a\rfloor} \cdot h_{ref,s}^{a\{(m-2)/a\}}). \tag{4.29}$$

holds on X_s. Here we have neglected the effect of the singularities $h_{D,s}$ by the choice of a and A. Apparently $\tilde{h}_{m-1}$ has a semipositive curvature current on U. Hence $\tilde{h}_{m-1} \cdot h_D$ is a singular hermitian metric on $(A + m(K_X + D) - K_X)|U$ with semipositive curvature current. Then by the L^2-extension theorem ([O-T, O]), we may extend every element of

$$\Gamma(X_s, \mathcal{O}_{X_s}(A_s + m(K_{X_s} + D_s)) \otimes \mathcal{I}(\tilde{h}_{m-1} \cdot h_D|X_s))$$

to an element of

$$\Gamma(f^{-1}(U), \mathcal{O}_X(A + m(K_X + D)) \otimes \mathcal{I}(\tilde{h}_{m-1} \cdot h_D)).$$

And by (4.29), we have that

$$\tilde{h}_{m-1} \cdot h_D|X_s = O(h_{A,s} \cdot \hat{h}_{can,s}^{a\lfloor(m-2)/a\rfloor} \cdot h_{ref,s}^{a\{(m-2)/a\}} \cdot h_{D,s}). \tag{4.30}$$

We note that by the choice of A, $\mathcal{O}_{X_s}(A_s + m(K_{X_s} + D_s)) \otimes \mathcal{I}(\hat{h}_{can,s}^{a\lfloor(m-1)/a\rfloor} \cdot h_{D,s})$ is globally generated over X_s. Hence by (4.30) we may extends every element of $H^0(X_s, \mathcal{O}_{X_s}(A_s + m(K_{X_s} + D_s)) \otimes \mathcal{I}(\hat{h}_{can,s}^{a\lfloor(m-1)/a\rfloor} \cdot h_{D,s}))$ to an element ot $\Gamma(f^{-1}(U), \mathcal{O}_X(A + m(K_X + D)))$ and the estimate

$$\tilde{h}_m|X_s = O(h_{A,s} \cdot \hat{h}_{can,s}^{a\lfloor(m-1)/a\rfloor} \cdot h_{ref,s}^{a\{(m-1)/a\}}) \tag{4.31}$$

holds by the same argument as above. Hence by induction on m, we see that (4.31) holds for every $m \geqq 1$. (4.31) implies the inclusion:

$$H^0(X_s, \mathcal{O}_{X_s}(A_s + m(K_{X_s} + D_s)) \otimes \mathcal{I}(\hat{h}_{can,s}^{a\lfloor(m-1)/a\rfloor})) \hookrightarrow E_{m,s} \tag{4.32}$$

and the assertion (1) of Lemma 4.8 holds.

By using Hölder's inequality as (2.14) and the trivial inequality: $a\lfloor(m-1)/a\rfloor \leqq m-1$, we shall transform the inclusion (4.32) to the lower estimates of $\hat{K}_m^A|X_s$ (cf. (4.16)):

$$\hat{K}_m^A|X_s \geqq K\left(A_s + m(K_{X_s} + D_s), h_A \cdot \hat{h}_{can,s}^{m-1} \cdot h_{D,s}\right)^{\frac{1}{m}} \cdot \left(\int_{X_s} \hat{h}_{can,s}^{-1} \cdot h_{D,s}\right)^{-\frac{m-1}{m}}. \tag{4.33}$$

We note that by Lemma 2.3 and Remark 2.4, we see that

$$\limsup_{m\to\infty} h_{A,s}^{\frac{1}{m}} \cdot K(A_s + m(K_{X_s} + D_s), h_{A,s} \cdot \hat{h}_{can,s}^{m-1} \cdot h_{D,s})^{\frac{1}{m}} = \hat{h}_{can,s}^{-1}$$

holds. Then by the Hölder inequality, similarly as (2.13) and (2.14) we have the estimate:

$$\limsup_{m\to\infty} h_{A,s}^{\frac{1}{m}} \cdot \hat{K}_m^A|X_s \geqq \left(\int_{X_s} \hat{h}_{can,s}^{-1} \cdot h_{D,s}\right)^{-1} \cdot \hat{h}_{can,s}^{-1}. \tag{4.34}$$

Hence by setting

$$C := \int_{X_s} \hat{h}_{can,s}^{-1} \cdot h_{D,s},$$

we have the estimate:

$$\hat{h}_{can}|X_s \leqq C \cdot \hat{h}_{can,s} \tag{4.35}$$

This completes the proof of Lemma 4.8. $\square$

By Lemma 4.8 and Theorem 4.2, we see that $\hat{h}_{can}|X_s$ is an AZD on $K_{X_s} + D_s$ with minimal singularities. Since $s \in S^\circ$ is arbitrary, we complete the proof of the assertion (2) of Theorem 4.3.

Now we shall prove the assertion (3) of Theorem 4.3. For $\ell, m \geqq 1$, we set $E_m^{(\ell)} := f_*\mathcal{O}_X(\ell A + m(K_{X/S} + D))$. We note that $E_m^{(\ell)}$ is a vector bundle, because we have assumed that $\dim S = 1$. We set

$$F := \{s \in S^\circ | E_{m,s}^{(\ell)} \neq \Gamma(X_s, \mathcal{O}_{X_s}(\ell A_s + m(K_{X_s} + D_s)) \text{ for some } \ell, m \geqq 1\}, \tag{4.36}$$

where $E_{m,s}^{(\ell)}$ denotes the fiber at s. Then by the definitions of $\hat{h}_{can,s}$ and $\hat{h}_{can}$, we see that for every $s \in S^\circ \backslash F$,

$$\hat{h}_{can}|X_s \leqq \hat{h}_{can,s}$$

holds on X_s. To prove that F is empty, we need to prove that $h^0(X_s, \mathcal{O}_{X_s}(\ell A_s + m(K_{X_s} + D_s)))$ is locally constant on S° for every $\ell, m \geqq 1$. But this follows from Theorem 1.15. Here we note that to prove Theorem 1.15. we need to use only the assertions (1) and (2) of Theorem 4.3. See Section 4.8 below.

4.5 Proof of Theorem 4.3; Non log general type case

Next we shall consider the case that $K_{X_s} + D_s$ is not necessarily big and D is $\mathbb{Q}$-linearly equivalent to a genuine line bundle B. In this case we shall consider the perturbation: $K_{X/S} + D + \ell^{-1}A(\ell = 1, 2, \cdots)$. Since $K_{X/S} + D + \ell^{-1}A|X_s$ is big for every $s \in S^\circ$ by the assumption, we may apply the argument in the last subsection. But there is a minor difference that $K_{X/S} + D + \ell^{-1}A|X_s$ is not Cartier. This is by no means an essential difficulty. But we need to modify the argument in an obvious way. Then we let ℓ tend to infinity.

More precisely we argue as follows. Let $G = b(\ell)A$ be a positive multiple of A. We set

$$E_{\ell,m} := f_*\mathcal{O}_X(G + m(K_{X/S} + D) + \lfloor m/\ell \rfloor A) \tag{4.37}$$

and

$$\hat{K}_{\ell,m}^G(s) := \sup\left\{ |\sigma|^{\frac{2}{m}};\ \sigma \in E_{\ell,m,s},\ \parallel \sigma \parallel_{\ell,m,s} = 1 \right\}, \tag{4.38}$$

where

$$\parallel \sigma \parallel_{\ell,m,s} := \left| \int_{X_s} h_{A,s}^{\frac{b(\ell)+\lfloor m/\ell \rfloor}{m}} \cdot h_{D,s} \cdot (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right|^{\frac{m}{2}}. \tag{4.39}$$

We set

$$\hat{K}_{\ell,\infty}^G := \text{the upper envelope of } \limsup_{m\to\infty} \hat{K}_{\ell,m}^G \tag{4.40}$$

and

$$\hat{h}_{can,\ell,G} := \frac{1}{\hat{K}_{\ell,\infty}^G}. \tag{4.41}$$

And we set

$$\hat{h}_{can,\ell} := \text{the lower envelope of } \inf_G h_{can,\ell,G}, \tag{4.42}$$

where G runs all the positive multiples of A. Replacing $f : X \to S$ by $X_s \to \{s\}$ we obtain $\hat{h}_{can,\ell,G,s}$ and $\hat{h}_{can,\ell,s}$. By [B-C-H-M] there exists a modification

$$\mu_{\ell,s} : Y_{\ell,s} \to X_s$$

such that $\mu_{\ell,s}^*(K_{X_s} + D_s + \ell^{-1}A)$ has a Zariski decomposition:

$$\mu_{k,s}^*(K_{X_s} + D_s + \frac{1}{\ell}A) = P_{\ell,s} + N_{\ell,s}. \tag{4.43}$$

Let $a = a(\ell)$ be a positive integer such that $a \cdot P_{\ell,s}$ is Cartier. Here we shall use the same notation as before for simplicity.

Lemma 4.9 *If we take G (depending on ℓ) sufficiently ample, we have the followings.*

(1) For every positive integer m, every element of $\Gamma(X_s, \mathcal{O}_{X_s}(G_s + m(K_{X_s} + D_s) + \lfloor m/\ell \rfloor A)) \otimes \mathcal{I}(h_{D,s} \cdot \hat{h}_{can,\ell,s}^{a\lfloor (m-1)/a \rfloor}))$ extends to an element of $\Gamma\left(f^{-1}(U), \mathcal{O}_X(G + m(K_X + D) + \lfloor m/\ell \rfloor A)\right)$, where $G_s := G|X_s$ and $h_{D,s}$ denotes the restriction $h_D|X_s$.

(2) There exists a positive constant C independent of ℓ such that

$$\hat{h}_{can,\ell}|X_s \leqq C \cdot h_{A,s}^{\frac{1}{\ell}} \cdot \hat{h}_{can,s} \tag{4.44}$$

holds on X_s. □

Proof. The proof of the assertion (1) is parallel to that of Lemma 4.8, i.e., we use the successive extensions. The only difference here is that we need to tensorize (A, h_A) every ℓ-steps. Then as in Lemma 4.8, we see that we have that

$$\hat{h}_{can,\ell}|X_s = O(\hat{h}_{can,\ell,s}) \tag{4.45}$$

holds as (4.35).

Let us prove the assertion (2). By the assertion (1) we see that since $\hat{h}_{can,\ell}|X_s$ is an AZD with minimal singularities $\ell^{-1}A + (K_{X_s} + D_s)$ by (4.45) and $\hat{h}_{can,s}$ is an AZD with minimal singularities on $K_{X_s} + D_s$. Since A is ample it is clear that

$$\hat{h}_{can,\ell}|X_s \leqq O\left(h_{A,s}^{\frac{1}{\ell}} \cdot \hat{h}_{can,s}\right) \tag{4.46}$$

holds on X_s. Our task is to find a constant C independent of ℓ such that (4.44) holds. By (4.46) and the assertion (1), we have the inclusion:

$$H^0(X_s, \mathcal{O}_{X_s}(G_s + m(K_{X_s} + D_s) + \lfloor m/\ell \rfloor A) \otimes \mathcal{I}(\hat{h}_{can,s}^{a\lfloor (m-1)/a \rfloor} \cdot h_{D,s})) \hookrightarrow E_{\ell,m,s}. \tag{4.47}$$

Then we tranform the inclusion (4.47) to the inequality:

$$\hat{K}_{\ell,m}^{G}|X_s \geqq K\left(G_s + m(K_{X_s} + D_s) + \lfloor m/\ell \rfloor A_s, h_{A,s}^{b(\ell)+\lfloor m/\ell \rfloor} \cdot \hat{h}_{can,s}^{m-1} \cdot h_{D,s}\right)^{\frac{1}{m}} \cdot \left(\int_{X_s} \hat{h}_{can,s}^{-1} \cdot h_{D,s}\right)^{-\frac{m-1}{m}} \tag{4.48}$$

obtained just as (4.33) above. Hence letting m tend to infinity, by Lemma 2.3,

$$\hat{h}_{can,\ell}|X_s \leqq C \cdot \left(h_{A,s}^{\frac{1}{\ell}} \cdot \hat{h}_{can,s}\right)$$

holds for

$$C := \int_{X_s} \hat{h}_{can,s}^{-1} \cdot h_{D,s},$$

hence it is independent of ℓ. Hence the assertion (2) holds. □

Lemma 4.9 implies that

$$\hat{h}_{can,\infty} := \text{the lower envelope of } \liminf_{\ell \to \infty} \hat{h}_{can,\ell} \tag{4.49}$$

exists and $\hat{h}_{can,\infty}|X_s$ is an AZD on $K_{X_s} + D_s$ with minimal singularities (cf. Definition 5.2). In fact the upper estimate of $\hat{h}_{can,\infty}$ follows from the assertion (1) of Lemma 4.9 and the lower estimate follows from the same argument as in Section 2.1.

Hence if A is sufficiently ample, by the L^2-extension theorem ([O-T, O]), we see that for every $m \geqq 0$ every element of $H^0(X_s, \mathcal{O}_{X_s}(A_s + m(K_{X_s} + D_s)) \otimes \mathcal{I}(\hat{h}_{can,\infty}^{m-1} \cdot h_D|X_s))$ extends to an element of

$H^0(f^{-1}(U), \mathcal{O}_X(A+m(K_{X/S}+D)))$. Since $\hat{h}_{can,\infty}|X_s$ is an AZD on $K_{X_s}+D_s$ with minimal singularities as above, the inclusion:

$$H^0(X_s, \mathcal{O}_{X_s}(A_s + m(K_{X_s} + D_s)) \otimes \mathcal{I}(\hat{h}^{m-1}_{can,s} \cdot h_{D,s})) \subseteq \tag{4.50}$$

$$H^0(X_s, \mathcal{O}_{X_s}(A_s + m(K_{X_s} + D_s)) \otimes \mathcal{I}(\hat{h}^{m-1}_{can,\infty} \cdot h_D|X_s))$$

holds. Hence by the L^2-extension theorem ([O-T, O]), we may extend every element of $H^0(X_s, \mathcal{O}_{X_s}(A_s+ m(K_{X_s}+D_s))\otimes\mathcal{I}(\hat{h}^{m-1}_{can,s}\cdot h_{D,s}))$ to an element of $H^0(f^{-1}(U), \mathcal{O}_X(A+m(K_{X/S}+D)))$. Then as (4.33) we may transform the inclusion (4.50) to the inequality:

$$\hat{K}^A_m|X_s \geqq K\left(A_s + m(K_{X_s} + D_s), h_A \cdot \hat{h}^{m-1}_{can,s} \cdot h_{D,s}\right)^{\frac{1}{m}} \cdot \left(\int_{X_s} \hat{h}^{-1}_{can,s} \cdot h_{D,s}\right)^{-\frac{m-1}{m}},$$

holds and repeating the same estimate as above (see (4.34)), letting m tend to infinity, by Lemma 2.3, we see that

$$\hat{h}_{can}|X_s \leqq \left(\int_{X_s} \hat{h}^{-1}_{can,s} \cdot h_{D,s}\right) \cdot \hat{h}_{can,s}.$$

holds on X_s and $\hat{h}_{can}|X_s$ is an AZD on $K_{X_s}+D_s$ with minimal singularities. Since $s \in S^\circ$ is arbitrary, we complete the proof of the assertion (2) of Theorem 4.3. The rest of the proof (the proof of the assertion (3)) is completely parallel to that of the previous subsection. We complete the proof of Theorem 4.3, assuming the boundary D is $\mathbb{Q}$-linearly equivalent to a Cartier divisor.

4.6 Dynamical systems of singular hermitian metrics

In this subsection, we complete the proof of Theorem 4.3. Here we do not assume that the boundary B is $\mathbb{Q}$-linearly equivalent to a genuine line bundle. In Section 5.3, we also give an alternative proof by using the ideas in [E-P].

First we shall prove the following theorem. The technique used here is essentially the same as the Ricci iteration in [T8].

Theorem 4.10 *Let $f : X \longrightarrow S$ be a proper surjective projective morphism between complex manifolds with connected fibers and let D be an effective $\mathbb{Q}$-divisor on X such that the set:*

$$S^\circ := \{s \in S|\ f \textit{ is smooth over } s \textit{ and } (X_s, D_s) \textit{ is KLT}\}$$

is nonempty. Suppose that S is connected and for some $s_0 \in S^\circ$, $K_{X_{s_0}} + D_{s_0}$ is pseudoeffective.

Then the followings hold.

(1) $K_{X_s} + D_s$ is pseudoeffective for every $s \in S^\circ$.

(2) $K_{X/S} + D$ is pseudoeffective (cf. Definition 1.6). □

Remark 4.11 *Here the pseudoeffectivity is defined as Definition 1.6. Hence we do not assume the compactness of the base space S.* □

Proof. The proof is quite similar to that of Lemma 4.8. The only essential difference is that we need the double induction, because of D_s is not $\mathbb{Q}$-linearly equivalent to a Cartier divisor. First we may and do assume that $\dim S = 1$ without loss of generality. Let A be a sufficiently ample line bundle on X such that

$$L_0 := A + (q-1)(K_{X/S} + B) \tag{4.51}$$

is ample. Let us fix a C^∞-hermitian metric on h_{L_0} on L_0 with strictly positive curvature. Then we define a singular hermitian metric $K_{X/S} + B + L_0|X^\circ = A + q(K_{X/S} + B)|X^\circ$ by

$$h_0 := \hat{h}_{can}(L_0 + B, h_{L_0} \cdot h_D). \tag{4.52}$$

Here we have used the relative version of Theorem 4.6, i.e., we take the direct image $f_*\mathcal{O}_X(m(K_{X/S} + B + L_0))$ for every sufficiently divisible $m > 0$ and construct the metric just as in Thorem 1.12 by using the similar construction as in Theorem 4.6. Then h_0 is of semipositive curvature current over X° by using Theorem 3.1 as in Section 3.2 and it extends to a singular hermitian metric with semipositive curvature on $q(K_{X/S} + B) + A$ by the same argument as in Section 3.3. Now we set

$$L_1 := (q-1)(K_{X/S} + B) + \frac{q-1}{q}A \tag{4.53}$$

and define the singular hermitian metric h_1 on $K_{X/S} + B + L_1$ over X° by

$$h_1 := \hat{h}_{can}(L_1 + B, h_0^{\frac{q-1}{q}} \cdot h_D). \tag{4.54}$$

Similarly as h_0, h_1 is of semipositive curvature current over X° and it extends to a singular hermitian metric on $K_{X/S} + B + L_1$ with semipositive curvature current. Inductively for every positive integer m, we set

$$L_m := (q-1)(K_{X/S} + B) + \left(\frac{q-1}{q}\right)^m A \tag{4.55}$$

and

$$h_m := \hat{h}_{can}(L_m + B, h_{m-1}^{\frac{q-1}{q}} \cdot h_D). \tag{4.56}$$

Then by induction on m, using Theorem 3.1, we see that h_m has semipositive curvature for every m. The above inductive construction is not well defined apriori, since we do not assume the pseudoeffectivity of $(K_{X/S} + B)|X_s$ for every $s \in S^\circ$. But the well definedness of h_m can be verified by successive extensions as follows.

Let U be a neighborhood of s_0 in S° which is biholomorphic to the unit disk Δ in $\mathbb{C}$. We may assume that $s_0 = 0$ on $U \simeq \Delta$. Let us assume the followings:

(1) We have already defined the singular hermitian metric h_{m-1} on $K_{X/S} + B + L_{m-1}$ with semipositive curvature.

(2) $h_{m-1}|X_0$ is an AZD of $K_{X_0} + B_0 + L_{m-1}|X_0$ with minimal singularities.

These assumptions are certainly satisfied for $m - 1 = 0$, if we take A sufficiently ample (see (4.51) and (4.52)). Under these assumptions, we shall prove the followings:

$(\mathrm{A}0)_m$ $K_{X_s} + B_s + L_m|X_s$ is pseudoeffective for every $s \in S^\circ$. Hence the singular hermitian metric h_m on $K_{X/S} + B + L_m$ is well defined and has semipositive curvature.

$(\mathrm{B}0)_m$ $h_m|X_0$ is an AZD of $K_{X_0} + B_0 + L_m|X_0$ with minimal singularities.

Let H be a sufficiently ample line bundle on X in the sense of Proposition 5.1 and let h_H be a C^∞-hermitian metric on H with strictly positive curvature.

We shall construct the singular hermitian metric $\tilde{h}_{m,\ell}$ on $H + \lfloor \ell(K_{X/S} + B + L_m) \rfloor | f^{-1}(U)$ with semipositive curvature for every $\ell \geqq 0$ by induction on ℓ as follows. Let h_A be a C^∞-hermitian metric on A with strictly positive curvature.

For $\ell = 0$, we set $\tilde{h}_{m,0} := h_H$. Suppose that we have already constructed $\tilde{h}_{m,\ell-1}$ for some $\ell \geqq 1$. We shall extend every element of

$$H^0(X_0, \mathcal{O}_{X_0}(H + \lfloor \ell(K_{X/S} + B + L_m) \rfloor) \otimes \mathcal{I}(\tilde{h}_{m,\ell-1} \cdot h_{m-1}^{\frac{q-1}{q}} \cdot h_D|X_0)) \tag{4.57}$$

to an element of

$$H^0(f^{-1}(U), \mathcal{O}_X(H + \lfloor \ell(K_{X/S} + B + L_m) \rfloor) \otimes \mathcal{I}(\tilde{h}_{m,\ell-1} \cdot h_{m-1}^{\frac{q-1}{q}} \cdot h_D))$$

by the L^2-extension theorem ([O-T, O]). In fact we use the semipositively curved metric:

$$h_A^{\delta_\ell} \cdot \tilde{h}_{m,\ell-1} \cdot h_{m-1}^{\frac{q-1}{q}} \cdot h_D$$

on $H + \lfloor \ell(K_{X/S} + B + L_m) \rfloor - K_{X/S}$, where

$$\delta_\ell := \left\lfloor \ell \left(\frac{q-1}{q} \right)^m \right\rfloor - \left\lfloor (\ell - 1) \left(\frac{q-1}{q} \right)^m \right\rfloor$$

to apply the L^2-extension theorem. Extending a set of basis of $H^0(X_0, \mathcal{O}_{X_0}(H + \lfloor \ell(K_{X/S} + B + L_m) \rfloor) \otimes \mathcal{I}(\tilde{h}_{m,\ell-1} \cdot h_{m-1}^{\frac{q-1}{q}} \cdot h_D | X_0))$ by the L^2-extension theorem, we define the singular hermitian metric $\tilde{h}_{m,\ell}$ on $H + \lfloor \ell(K_{X/S} + B + L_m) \rfloor | f^{-1}(U)$ with semipositive curvature just as (4.27) above. Here we note that since (X_0, D_0) is KLT,

$$H^0(X_0, \mathcal{O}_{X_0}(H + \lfloor \ell(K_{X/S} + B + L_m) \rfloor) \otimes \mathcal{I}(\tilde{h}_{m,\ell-1} \cdot h_{m-1}^{\frac{q-1}{q}} \cdot h_D | X_0))$$

contains the subspace:

$$H^0_{(\infty)}(X_0, \mathcal{O}_{X_0}(H + \lfloor \ell(K_{X/S} + B + L_m) \rfloor), (\tilde{h}_{m,\ell-1} \cdot h_{m-1}^{\frac{q-1}{q}} \cdot h_A^{\delta_\ell}) | X_0 \cdot h_{B,0})$$

of $H^0(X_0, \mathcal{O}_{X_0}(H + \lfloor \ell(K_{X/S} + B + L_m) \rfloor))$ consisting the bounded holomorphic sections with respect to $(\tilde{h}_{m,\ell-1} \cdot h_{m-1}^{\frac{q-1}{q}} \cdot h_A^{\delta_\ell}) | X_0 \cdot h_{B,0}$, where $h_{B,0}$ is a C^∞-hermitian metric on $B | X_0$. Let $h_{m,0,min}$ be an AZD of $K_{X_0} + B_0 + L_m | X_0$ with minimal singularities (cf. Section 5.2). We shall use this fact for the estimate of $\tilde{h}_{m,\ell} | X_0$ as the use of (4.28) in the proof of Lemma 4.8. We note that

$$h_{m-1} | X_0 = O\left(h_{m,0,min} \cdot h_{A,0}^{\frac{1}{q}\left(\frac{q-1}{q}\right)^{m-1}} \right)$$

holds by the assumption that $h_{m-1} | X_0$ is an AZD of $K_{X_0} + B_0 + L_{m-1} | X_0$ with minimal singularities. Then using [B-C-H-M] again, as (4.31) in Lemma 4.8, by induction on ℓ, we see that

$$\tilde{h}_{m,\ell} | X_0 = O\left(h_{H,0} \cdot h_{m,0,min}^\ell \cdot h_{A,0}^{-\{\ell\left(\frac{q-1}{q}\right)^m\}} \right) \tag{4.58}$$

holds, where $h_{H,0} := h_H | X_0$ (Actually as in Lemma 4.8, we have a slightly better estimate). Hence by the sufficiently ampleness of H, $\{\tilde{h}_{m,\ell}\}_{\ell=0}^\infty$ is well defined on U. In particular $K_{X/S} + B + L_m$ is pseudoeffective on $f^{-1}(U)$ and h_m is well defined on $f^{-1}(U)$ because the pseudoeffectivity on the fiber is closed under specialization over S°. We transform (4.58) into the estimate:

$$h_m | X_0 = O(h_{m,0,min}) \tag{4.59}$$

as (4.35) by the same argument as (4.31) $\sim$ (4.35). And we see that $h_m | X_0$ is an AZD with minimal singularities on $K_{X_0} + B_0 + L_m | X_0$. Hence the induction works. In this way, $\{\tilde{h}_{m,\ell}\}_{\ell=0}^\infty$ is well defined for every $m \geqq 0$. And this implies that $(K_{X/S} + B + L_m) | f^{-1}(U)$ is pseudoeffective for every $m \geqq 0$. Letting m tend to infinity, we see that $K_{X_s} + B_s$ is pseudoeffective for every $s \in U$. The openness of the pseudoeffectivity of $K_{X_s} + B_s + L_m | X_s$ is obtained just by repeating the above argument. Hence for every $s \in S^\circ$, $K_{X_s} + B_s + L_m | X_s$ is pseudoeffective because the pseudoeffectivity is closed under specializations. This implies that h_m is well defined and is a singular hermitian metric on $K_{X/S} + B + L_m$ with semipositive curvature over $X^\circ := f^{-1}(S^\circ)$ by Theorem 3.1 and induction on m. And it extends to a singular hermitian metric with semipositive curvature just as in Sections 3.2 and 3.3. By the semipositivity of the curvature of h_m we see that $K_{X/S} + B + L_m$ is pseudoeffective.

Replacing $0 = s_0$ by an arbitrary $s \in S^\circ$ and repeating the above argument, we prove the followings by induction on m.

$(\mathrm{A})_m$ $K_{X/S} + B + L_m$ is pseudoeffective on X and $K_{X_s} + B_s + L_m | X_s$ is pseudoeffective for every $s \in S^\circ$.

(B)$_m$ $h_m|X_s$ is an AZD with minimal singularities on $K_{X_s} + B_s + L_m|X_s$ for every $s \in S^\circ$.

Hence h_m is well defined and has semipositive curvature. Letting m tend to infinity, we see that $K_{X/S}+B$ is pseudoeffective and $K_{X_s} + B_s$ is pseudoeffective for every $s \in S^\circ$. $\square$

Now we shall complete the proof of Theorem 4.3. We shall use the same notation as above. In the above proof of Theorem 4.10, we have seen that $h_m|X_s$ is an AZD with minimal singularities on $K_{X_s} + B_s + L_m|X_s$. Hence again similarly as Lemma 4.9, (2), by the induction on m, we see that there exists a positive constant C independent of m such that for every $m \geqq 1$,

$$h_m|X_s \leqq \exp\left(C \cdot \sum_{k=0}^{m-1}\left(\frac{q-1}{q}\right)^k\right) \cdot h_{A,s}^{\left(\frac{q-1}{q}\right)^m} \cdot \hat{h}_{can,s}^q \tag{4.60}$$

holds. Letting m tend to infinity, we see that

$$\liminf_{m\to\infty}\left(h_A^{-\left(\frac{q-1}{q}\right)^m} \cdot h_m\right)|X_s \leqq \exp(C\cdot q)\cdot \hat{h}_{can,s}^q \tag{4.61}$$

holds. Since the lower estimate of the left-hand side is obtained as in Section 2.1, the lower semicontinuous envelope of the left-hand side is a well defined singular hermitian metric with semipositive curvature. We note that

$$\hat{h}_{can}^q|X_s = O\left(\liminf_{m\to\infty}\left(h_A^{-\left(\frac{q-1}{q}\right)^m} \cdot h_m\right)|X_s\right) \tag{4.62}$$

holds , since $\hat{h}_{can}$ is an AZD with minimal singularities on $K_{X/S} + D$ and $\liminf_{m\to\infty}\left(h_A^{-\left(\frac{q-1}{q}\right)^m} \cdot h_m\right)$ is a singular hermitian metric on $K_{X/S} + D$ with semipositive curvature. Combining (4.61) and (4.62), we see that

$$\hat{h}_{can}|X_s = O(\hat{h}_{can,s})$$

holds. This completes the proof of the asseretion (2) in Theorem 4.3. The rest of the proof is identical to the one of Theorem 1.12. $\square$

4.7 Variation of supercanonical AZD's for relative adjoint line bundles of KLT $\mathbb{Q}$-line bundles

We can generalize Theorem 4.3 to the case of the family of relative adjoint bundles of KLT singular hermitian line bundles.

Theorem 4.12 *Let $f : X \longrightarrow S$ be a proper surjective projective morphism between complex manifolds with connected fibers and let (L, h_L) be a pseudoeffective singular hermitian $\mathbb{Q}$-line bundle on X such that for a general fiber X_s, $(L, h_L)|X_s$ is KLT(cf. Definition 1.18). We set*

$$S^\circ := \{s \in S|\ f \text{ is smooth over } s \text{ and } (L,h_L)|X_s \text{ is well defined and KLT }\}.$$

Then there exists a singular hermitian metric $\hat{h}_{can}(L, h_L)$ on $K_{X/S} + L$ such that

(1) $\hat{h}_{can}(L, h_L)$ has semipositive curvature current,

(2) $\hat{h}_{can}(L, h_L)|X_s$ is an AZD on $K_{X_s} + L_s$ (with minimal singularities) for every $s \in S^\circ$,

(3) For every $s \in S^\circ$, $\hat{h}_{can}(L, h_L)|\,X_s \leqq \hat{h}_{can}((L, h_L)|X_s)$ holds, where $\hat{h}_{can}((L, h_L)|X_s)$ denotes the supercanonical AZD on $K_{X_s} + L_s$ with respect to $h_L|X_s$ (cf. Theorem 4.6). And $\hat{h}_{can}(L, h_L)|X_s = \hat{h}_{can}((L, h_L)|X_s)$ holds outside of a set of measure 0 on X_s for almost every $s \in S^\circ$. $\square$

The proof of Theorem 4.12 is completely parallel to the one of Theorem 4.3 above.

Assume that for every positive rational number ε, $L_s + \varepsilon A|X_s (s \in S^\circ)$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor $D_{\varepsilon,s}$ such that $(X_s, D_{\varepsilon,s})$ is KLT. Then we may apply [B-C-H-M] as in Section 4.5 and can prove Theorem 4.12 similarly as Theorem 4.3. To assure this assumption, we need the following lemma.

Lemma 4.13 *Let (F, h_F) be a pseudoeffective singular hermitian $\mathbb{Q}$-line bundle on a smooth projective variety M such that the curvature $\sqrt{-1}\,\Theta_{h_F}$ dominates a C^∞-Kähler form on M. Suppose that (F, h_F) is KLT (cf. Definition 1.18). Then there exists an effective $\mathbb{Q}$-divisor V on M such that L is $\mathbb{Q}$-linearly equivalent to V and (M, V) is KLT.* $\square$

Proof. By the assumption and Nadel's vanishing theorem ([N, p.561]), we see that for every sufficiently large m such that mF is Cartier, $\mathcal{O}_M(mF) \otimes \mathcal{I}(h_F^m)$ is globally generated. Take such a sufficiently large m and let σ be a general nonzero element of $H^0(M, \mathcal{O}_M(mF) \otimes \mathcal{I}(h_F^m))$ and set $V = m^{-1}(\sigma)$, where (σ) denotes the divisor associated with σ. Then (M, V) is KLT by the global generation property. $\square$

The rest of the proof of Theorem 4.12 is parallel to the one of Theorem 4.3. Hence we omit it. In fact we just need to replace $h_{D,s}$ by $h_{L,s} := h_L|X_s$. $\square$

The following pseudoeffectivity theorem is similar to [B-P, Theorem 0.1]. The advantage is that we deal with $\mathbb{Q}$-line bundles and without assuming the existence of sections on the special fiber. But our theorem has the additional KLT assumption.

Theorem 4.14 *Let $f : X \longrightarrow S$ be a proper surjective projective morphism between complex manifolds with connected fibers and let L be a $\mathbb{Q}$-line bundle on X with a singular hermitian metric h_L with semipositive curvature. We assume that S is quasiprojective or Stein. Suppose that the set:*

$$S^\circ := \{s \in S |\, f \text{ is smooth over } s,\ (L, h_L)|X_s \text{ is well defined and KLT}\}$$

is nonempty and there exists some $s_0 \in S^\circ$ such that $K_{X_{s_0}} + L|X_{s_0}$ is pseudoeffective.

Then $K_{X/S} + L$ is pseudoeffective on X. $\square$

The proof of Theorem 4.14 is parallel to that of Theorem 4.10, if we use the perturbation as in Section 4.5 and Lemma 4.13. Hence we omit it. The assumption that S is quasiprojective or Stein is used to globalize Lemma 4.13 on X.

4.8 Proof of Theorems 1.15 and 1.19

In this subsection we shall prove Theorems 1.15 and 1.19.

Let $f : X \longrightarrow S$ be a proper surjective projective morphism between complex manifolds with connected fibers. Let D be an effective $\mathbb{Q}$-divisor on X such that

(a) D is $\mathbb{Q}$-linearly equivalent to a $\mathbb{Q}$-line bundle B,

(b) The set: $S^\circ := \{s \in S |\, f$ is smooth over s and (X_s, D_s) is KLT $\}$ is nonempty.

If $K_{X_{s_0}} + D_{s_0}$ is pseudoeffective for some $s_0 \in S^\circ$, then for every $s \in S^\circ$, $K_{X_s} + D_s$ is pseudoeffective by Theorem 4.10. Hence we may and do assume for every $s \in S^\circ$, $K_{X_s} + D_s$ is pseudoeffective. In fact otherwise we see that $P_m(X_s, B_s)$ is identically 0 on S° for every $m \geqq 1$.

Since the problem is local, to prove Theorem 1.15, we may and do assume that S is the unit open disk in $\mathbb{C}$ and $S^\circ = S$. Let $\hat{h}_{can}$ be the relative supercanonical AZD on $f : (X, D) \to S$@ as in Theorem 4.3. Let m be an arbitrary positive integer such that mB is a genuine line bundle. Let $s \in S^\circ$ and let $\sigma \in H^0(X_s, \mathcal{O}_{X_s}(m(K_{X_s} + B_s)))$ be an arbitrary nonzero element. Since $\hat{h}_{can}|X_s$ is an AZD with minimal singularities (see Definition 5.2) by Theorems 4.2 and 1.13 (or Lemma 4.8),

$$\hat{h}_{can}|X_s = O\left(\frac{1}{|\sigma|^{\frac{2}{m}}}\right)$$

holds. Hence $\hat{h}^m_{can}(\sigma,\sigma)$ is bounded on X_s. We note that

$$|\sigma|^2 \cdot (\hat{h}^{m-1}_{can}|X_s) \cdot h_{D,s} = |\sigma|^2 \cdot (\hat{h}^m_{can}|X_s) \cdot ((\hat{h}^{-1}_{can}|X_s) \cdot h_{D,s})$$

holds and $(\hat{h}^{-1}_{can}|X_s) \cdot h_{D,s}$ is a locally integrable singular volume form on X_s, since (X_s, D_s) is KLT. Hence we see that

$$\int_{X_s} |\sigma|^2 \cdot (\hat{h}^{m-1}_{can}|X_s) \cdot h_{D,s}$$

is bounded. By Theorem 1.13 and the L^2-extension theorem ([O-T, O]), we may extend σ to an element of $H^0(X, \mathcal{O}_X(m(K_X + B)))$. Since $s \in S^\circ$ is arbitrary, noting the upper-semicontinuity theorem for cohomologies, we see that $P_m(X_s, B_s) = \dim H^0(X_s, \mathcal{O}_{X_s}(m(K_{X_s} + B_s)))$ is locally constant over S°. $\square$

The proof of Theorem 1.19 is simlar to the one of Theorem 1.15. Hence we omit it.

4.9 Semipositivity of the direct image of pluri log canonical systems

The semipositivity of the direct image of the relative pluricanonical system has been studied in many papers such as [F, Ka1, V1, V2]. But in the case of the relative pluri log canonical systems, not so much is known except [Ka3, p.175,Theorem 1.2].

Let $f : X \to S$ be a proper projective morphism between complex manifolds with connected fibers. Let D be an effective $\mathbb{Q}$-divisor on X such that $S^\circ := \{s \in S|\ f$ is smooth over s and (X_s, D_s) is KLT $\}$ is nonempty. Suppose that D is $\mathbb{Q}$-linearly equivalent to a $\mathbb{Q}$-line bundle B. Let m be a positive integer such that mB is a genuine line bundle. Then by Theorem 1.15, the direct image:

$$F_m := f_*\mathcal{O}_X(m(K_{X/S} + B)) \tag{4.63}$$

is locally free over S°. By Theorem 4.3, the relative supercanonical AZD $\hat{h}_{can}$ exists on $K_{X/S} + B$ and has semipositive curvature current. We define the metric h_m on $F_m|S^\circ$ by

$$h_m(\sigma, \sigma') := (\sqrt{-1})^{n^2} \int_{X_s} \hat{h}^{m-1}_{can} \cdot h_D \cdot \sigma \wedge \overline{\sigma'} \qquad (\sigma, \sigma' \in F_{m,s}), \tag{4.64}$$

where h_D is the metric defined as (4.15) and $n := \dim X - \dim S$. Then by Theorem 4.3 and [B-P, Theorem 3.5], we have the following theorem.

Theorem 4.15 *The locally bounded metric h_m on $F_m|S^\circ$ is semipositive in the sense h_m gives a singular hermitian metric with semipositive curvature on the tautological line bundle $\mathcal{O}(1)$ on $\mathbb{P}(F^*_m|S^\circ)$.* $\square$

Remark 4.16 *It is trivial to generalize Theorem 4.15 in the case of the direct image of the multi adjoint line bundle of a generically KLT line bundles.*

If Conjecture 2.16 is true and $\dim S = 1$, it is not difficult to see that h_m gives a singular hermitian metric with semipositive curvature on the tautological line bundle $\mathcal{O}(1)$ on the whole $\mathbb{P}(F^*_m)$. $\square$

For the different treatments such as weak semistability of the direct images of pluri log canonical systems, see [T7, T8, T9]. In these papers, the canonical metric comes from a log canonical bundle on the base space of an Iitaka fibrations and the construction of the metric is quite different from the one here.

5 Appendix

Here we collect miscellaneous facts.

5.1 Choice of the sufficiently ample line bundle A

In this subsection, we shall prove the following proposition.

Proposition 5.1 *Let X be a smooth projective n-fold. Then there exists an ample line bundle A on X such that for every pseudoeffective singular hermitian line bundle (L, h_L) on X, $\mathcal{O}_X(A + L) \otimes \mathcal{I}(h_L)$ and $\mathcal{O}_X(K_X + A + L) \otimes \mathcal{I}(h_L)$ are globally generated.* $\square$

Proof. We construct such an A by using L^2-estimates. In fact let g be a Kähler metric on X and for every x, d_x denotes the distance function from x and let $R > 0$ denotes the infimum of the injective radius on (X, g). Let ρ be a C^∞-function on $[0, R)$ such that

(1) $0 \leqq \rho \leqq 1$,

(2) $\operatorname{Supp} \rho \subset [0, \frac{2}{3}R]$,

(3) $\rho \equiv 1$ on $[0, \frac{1}{3}R]$.

Then we may take an ample line bundle A and a C^∞-hermitian metric h_A such that $\sqrt{-1}\left(\Theta_{h_A} + 2n\partial\bar{\partial}\left(\rho(d_x) \cdot \log d_x\right)\right)$ and $\mathrm{Ric}_g + \sqrt{-1}\left(\Theta_{h_A} + 2n\partial\bar{\partial}\left(\rho(d_x) \cdot \log d_x\right)\right)$ are closed strictly positive $(1,1)$ current on X for every $x \in X$. Then by Nadel's vanishing theorem [N, p.561], for every pseudoeffective singular hermitian line bundle (L, h_L) on X, $\mathcal{O}_X(A+L) \otimes \mathcal{I}(h_L)$ and $\mathcal{O}_X(K_X+A+L) \otimes \mathcal{I}(h_L)$ are globally generated. $\square$

Let h_A be a a C^∞-hermitian metric on A with strictly positive curvature as above. Let us fix a C^∞-volume form dV on X. By the L^2-extension theorem ([O]) we take a sufficiently ample line bundle A so that for every $x \in X$ and for every pseudoeffective singular hermitian line bundle (L, h_L), there exists a bounded interpolation operator:

$$I_x : A^2(x, (A+L)_x, h_A \cdot h_L, \delta_x) \to A^2(X, A+L, h_A \cdot h_L, dV)$$

such that the operator norm of I_x is bounded by a positive constant independent of x and (L, h_L), where $A^2(X, A+L, h_A \cdot h_L, dV)$ denotes the Hilbert space defined by

$$A^2(X, A+L, h_A \cdot h_L, dV) := \left\{ \sigma \in \Gamma(X, \mathcal{O}_X(A+L) \otimes \mathcal{I}(h_L)) \middle| \int_X \mid \sigma \mid^2 \cdot h_A \cdot h_L \cdot dV < +\infty \right\}$$

with the L^2-inner product:

$$(\sigma, \sigma') := \int_X \sigma \cdot \overline{\sigma'} \cdot h_A \cdot h_L \cdot dV$$

and $A^2(x, (A+L)_x, h_A \cdot h_L, \delta_x)$ is defined similarly, where δ_x is the Dirac measure supported at x. We note that if $h_L(x) = +\infty$, then $A^2(x, (A+L)_x, h_A \cdot h_L, \delta_x) = 0$.

5.2 Analytic Zariski decompositions and singular hermitian metrics with minimal singularities

In this paper we have used the notion of AZD's (cf. Definition 1.1). We note that there is a similar but **different** notion : singular hermitian metrics with minimal singularities introduced in [D-P-S] (see Definiton 5.2 below). I would like to explain the difference of these two notions here.

According to [D-P-S], an AZD is constructed for any pseudoeffective line bundle L as follows. Let h_L be any C^∞-hermitian metric on L. Let h_0 be an AZD on K_X defined by the lower envelope of :

$$\inf \left\{ h \mid h \text{ is a singular hermitian metric on } L \text{ with } \sqrt{-1}\,\Theta_h \geqq 0, h \geqq h_L \right\},$$

where the inf denotes the pointwise infimum. This construction is exactly the same as (2.5) above. Then by the classical theorem of Lelong ([L, p.26, Theorem 5]) it is easy to verify that h_0 is an AZD on L (cf. [D-P-S, Theorem 1.5]). By the definition, h_0 is of minimal singularities in the following sense.

Definition 5.2 *Let L be a pseudoeffective line bundle on a smooth projective variety X. An AZD h on L is said to be* **a singular hemitian metric with minimal singularities** *or* **an AZD with minimal singularities**, *if for any singular hermitian metric h' on L with semipositive curvature current, there exists a positive constant C such that*

$$h \leqq C \cdot h'$$

holds on X. In particular for any AZD h' on L the above inequality holds for some positive constant C. □

We note that any AZD's with minimal singularities are quasi-isometric, i.e., any two AZD's with minimal singularities h_1, h_2 on a common line bundle L, there exists a positive constant $C > 1$ such that

$$C^{-1} \cdot h_2 \leqq h_1 \leqq C \cdot h_2$$

holds. In particular for any AZD with minimal singularities h on a line bundle L, the multiplier ideal $\mathcal{I}(h^m)$ is uniquely determined for every m. And the above construction of an AZD is very easy. In the above sense, the AZD with minimal singularities is very canonical.

But in general, an AZD is not with minimal singularities as follows.

Example 5.3 *Let X be a smooth projective variety and let D be a divisor with simple normal crossings on X. Suppose that $K_X + D$ is ample. Then there exists a complete Kähler-Einstein form ω_E on $X\backslash D$ with $-\mathrm{Ric}_{\omega_E} = \omega_E$ and ω_E extends to a closed positive current on X with vanishing Lelong numbers and $[\omega_E] = 2\pi c_1(K_X + D)$ ([Ko]). The metric $h := (\omega_E^n)^{-1}(n = \dim X)$ is a singular hermitian metric on $K_X + D$ with strictly positive curvature on X. Let $D = \sum_i D_i$ be the irreducible decomposition of D and let σ_i be a nontrivial global section of $\mathcal{O}_X(D_i)$ with divisor D_i. h is an AZD on $K_X + D$, but h has logarithmic singularities along D, i.e., there exists a C^∞-hermitian metric on h_0 on $K_X + D$ such that*

$$h = h_0 \cdot \prod_i |\log \parallel \sigma_i \parallel|^2,$$

where $\parallel \sigma_i \parallel$ denotes the hermitian norm of σ_i with respect to a C^∞-hermitian metric on $\mathcal{O}_X(D_i)$ respectively. Hence h blows up along D. In particular h is not of minimal singularities. □

As above, even in the case of ample line bundles, some natural AZD's are not of minimal singularities. Indeed the notion of AZD's is much broader than the notion of singular hermitian metrics with minimal singularities. Much more general singular Kähler-Einstein metrics on LC pairs of log general type was considered in [T8]. More precisely in the paper, we have considered singular Kähler-Einstein metrics on LC pairs (X, D) of log general type such that the inverse of the Kähler-Einstein volume form is an AZD on $K_X + D$. In that case the AZD is not necessarily of minimal singularities as is seen in the above examples.

And also it is not clear whether the canonical AZD h_{can} defined in Section 1.1 has minimal singularities.

The above examples indicate us that we had better not to restrict ourselves to consider AZD's with minimal singularities to consider broader canonical singular hermitian metrics.

5.3 An alternative proof of Theorem 4.3

In this subsection, we shall give an alternative proof of Theorem 4.3 by using the argument in [E-P]. Here we assume the results in Sections 4.4 and 4.5. The reason why we present an alternative proof is that although the proof itself is far more complicated than the one in Section 4.6, it may indicate the way how to handle the extension without assuming the bigness. Hence it may have an independent interest.

The strategy of the proof is as follows. We follows the argument in [E-P] when the fibers over S° is of log general type. The key point of the proof is we subdivide the extension into several steps by using the logarithmic vanishing theorem as [E-P, Theorems 2.9, 3.2] for the extension of holomorphic sections similar to [E-P, Propositions 4.1 and 4,2]. But the theorem requires the bigness of the line bundle on

every component of the log canonical centers. In our case this condition need not be satisfied. Hence we perturb the log canonical bundle by adding ample $\mathbb{Q}$-line bundles as in Section 4.5. We note that this condition is stated in [E-P] by the language of augumented base locus B_+ (cf. [E-P]). Then as in Section 4.5, we take the limit. Since we have already known how to transform the inclusion of the multiplier ideals into the estimate of the canonical singular hermitian metric as Lemmas 4.8 and 4.9, this part is essentially nothing new. Hence the proof here is just a combination of the perturbation and the argument in [E-P] using the estimate of canonical singular hermitian metrics in Sections 4.4 and 4.5. Since we follow the argument in [E-P], we do not repeat the proof, when we just borrow the argument in [E-P].

Let us assume that D is $\mathbb{Q}$-linearly equivalent to a $\mathbb{Q}$-line bundle B. As for the proof of the assertion (1) of Theorem 4.3, nothing changes and it follows from Theorem 3.1. Hence we only need to verify the assertions (2) and (3) of Theorem 4.3. For simplicity we shall consider the case: $\dim S = 1$. To prove the assertion (3), we may and do assume that $S = S^\circ = \Delta$ hold.

Lemma 5.4 *For every $s \in S^\circ$, there exists a positive constant C_+ depending on s such that*

$$\hat{h}_{can}|X_s \leqq C_+ \cdot \hat{h}_{can,s}$$

holds on X_s. $\square$

Proof. Let U be an open neighborhood of s in S° which is biholomorphic to the unit open polydisk $\Delta^k(k = \dim S)$ as above. We shall identify $K_{X/S}|f^{-1}(U)$ with K_X by

$$\otimes f^*dt : K_{X/S} \longrightarrow K_X, \tag{5.1}$$

where t is the standard coordinate on Δ. Let A be an ample $\mathbb{Q}$-line bundle on X. By [B-C-H-M] the relative log canonical ring $\oplus_{m=0}^{\infty} f_*\mathcal{O}_S(\lfloor m(K_{X/S} + A + D)\rfloor)$ is locally finitely generated. Let q be a positive integer such that $q(K_{X/S} + A + D)$ is integral. In this case the relative log canonical ring is big at s, i.e., the image of

$$\oplus_{m=0}^{\infty} f_*\mathcal{O}_X(mq(K_{X/S} + A + D))_s \to \oplus_{m=0}^{\infty} H^0(X_s, \mathcal{O}_{X_s}(mq(K_{X_s} + A_s + D_s))) \tag{5.2}$$

is a subring of maximal growth. Let $\mu_s : Y_s \to X_s$ be a modification such that Zariski decomposition of the image of (5.2):

$$\mu_s^*(K_{X_s} + A_s + B_s) = P_s + N_s \quad (P_s, N_s \in \mathrm{Div}(Y_s) \otimes \mathbb{Q}) \tag{5.3}$$

exists as (4.22), i.e., P_s is nef, N_s is effective and $H^0(Y_s, \mathcal{O}_{Y_s}(\lfloor mqP_s \rfloor))$ is isomorphic to the image of $f_*\mathcal{O}_X(mq(K_{X/S} + D))_s \to H^0(X_s, \mathcal{O}_{X_s}(mq(K_{X_s} + B_s)))$. In this case P_s is semiample by the finite generation of the relative log canonical ring. Since P_s is semiample, and (X_s, D_s) is KLT, again by [B-C-H-M], there exists a Zariski decomposition

$$\mu_s^*(K_{X_s} + A + B_s) = Q_s + E_s \quad (Q_s, E_s \in \mathrm{Div}(Y_s) \otimes \mathbb{Q}) \tag{5.4}$$

of $\mu_s^*(K_{X_s} + B_s)$ such that Q_s is semiample. Hereafter for simplicity, we shall assume that $D_{red} + X_s$ is a simple normal crossing divisor. The general case is handled by taking a log resolution of $(X, D + X_s)$ as in [E-P, Proposition 5.4]. We shall review the notion of adjoint ideals in [E-P].

Definition 5.5 *(Adjoint ideals ([E-P])) Let Γ be a reduced simple normal crossing divisor on X and $\mathfrak{a} \subset \mathcal{O}_X$ an ideal sheaf such that no log-canonical center of Γ is contained in $Z(\mathfrak{a})$. Let $f : Y \to X$ be a common log-resolution for the pair (X, Γ) and the ideal $\mathfrak{a}$, ad write $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E)$. We set*

$$\mathrm{Adj}_\Gamma(X, \mathfrak{a}^\lambda) := f_*\mathcal{O}_Y(K_{Y/X} - f^*\Gamma + \sum_{\mathrm{ld}(\Gamma, D_i) = 0} D_i - \lfloor \lambda E \rfloor), \tag{5.5}$$

where the sum appearing in the expression is taken over all divisors on Y having log-discrepancy 0 with respect to Γ, i.e., among those appearing in Γ' in the expression $K_Y + \Gamma' = f^*(K_X + \Gamma)$. We note that

$$\mathrm{Adj}_\Gamma(X, \mathfrak{a}^\lambda) \subset \mathcal{I}(X, \mathfrak{a}^\lambda).$$

For a graded system $\mathfrak{a}_* = \{\mathfrak{a}_m\}$ of ideal sheaves,we set

$$\mathrm{Adj}_\Gamma(X, \mathfrak{a}_*^\lambda) := \mathrm{Adj}_\Gamma(X, \mathfrak{a}_m^{\lambda/m})$$

for m sufficiently divisible. For a $\mathbb{Q}$-effective line bundle L, we set

$$\mathrm{Adj}_\Gamma(X, \| L \|) := \mathrm{Adj}_\Gamma(X, \mathfrak{b}_*),$$

where $\mathfrak{b}_*$ denotes the graded system of the base ideal of L. We call $\mathrm{Adj}_\Gamma(X, \| L \|)$ the asymptotic adjoint ideal of L with respect to Γ. These definitions can be generalized to the case of a pair (X, Λ) of X and an effective $\mathbb{Q}$-divisor Λ provided $\mathbf{B}(L) \cup \mathrm{Supp}\,\Lambda$ does not contain any LC center of Γ, where $\mathbf{B}(L)$ denotes the stable base locus of L. $\square$

Let us interpret Definition 5.5 in terms of singular hermitian metrics. To do this we introduce the following notion.

Definition 5.6 *Let (L, h_L) be a singular hermitian line bundle on a complex manifold W. We say that h_L is of algebraic singularities, if h_L is written locally as*

$$h_L = h_0 \cdot \left(\sum_{i=1}^{N} |f_i|^2 \right)^{-\alpha},$$

where $f_1, \cdots, f_N$ are local holomorphic functions, h_0 is a local C^∞-hermitian metric on L and α is a positive number. $\square$

Let (L, h_L) be a singular hermitian line bundle of algebraic singularities. Then there exists a modification $\mu : V \to W$ such that

(1) The exceptional divisor of μ is a simple normal crossing divisor,

(2) There exists an effective $\mathbb{R}$-divisor E on V such that $\mathrm{Supp}\,E$ is a simple normal crossing divisor on V and $\mathcal{I}(h_L^m) = \mu_*\mathcal{O}_V(K_{V/W} - \lfloor mE \rfloor)$ holds for every positive integer m.

Let Γ be a reduced simple normal crossing divisor on W. By using this divisor E, we can define the adjoint ideal $\mathrm{Adj}_\Gamma(W; h_L)$ as (5.5), if the singular set $\mathrm{Sing}\,h_L$ of h_L does not contain any components of Γ. This notion is used to rewrite the argument in [E-P] in terms of singular hermitian metrics and to avoid the use of asymptotic multiplier ideals in [E-P]. For a pseudoeffective singular hermitian line bundle (L, h_L) on a smooth projective variety W, we say that (L, h_L) is big, if $\limsup_{m\to\infty} m^{-\dim W} \cdot h^0(W, \mathcal{O}_W(mL) \otimes \mathcal{I}(h_L^m))$ is positive. The following proposition follows from the same argument as in the proof of [E-P, Theorems 2.9 and 3.2]. This is nothing but the singular hermitian version of Norimatsu's vanishing theorem ([No]).

Proposition 5.7 *Let X be a smooth projective variety and let Γ be a reduced simple normal crossing divisor on X. Let (L, h_L) be a singular hermitian line bundle on X such that*

(1) h_L is of algebraic singularities,

(2) (L, h_L) is big and the restriction $(L, h_L)|\Gamma_j$ to every irrducible component Γ_j of Γ is well defined and big.

Then

$$H^q(X, \mathcal{O}_X(K_X + L + \Gamma) \otimes \mathrm{Adj}_\Gamma(X, h_L)) = 0$$

holds for every $q \geqq 1$. $\square$

The following extension theorem is similar to [E-P, Proposition 4.2]. The proof follows from Proposition 5.7.

Proposition 5.8 *Let X be a smooth projective variety and $S \subset X$ be a smooth divisor. Let Γ be an effective integral divisor on X such that $S+\Gamma$ is a reduced simple normal crossing divisor. Let $\Gamma = \sum \Gamma_j$ be the irreducible decomposition. Let (L, h_L) be a big pseudoeffetive line bundle on X with algebraic singularities, such that no log-canonical center of $(X, S+\Gamma)$ is contained in* $\mathrm{Sing}\, h_L \cup \mathrm{Supp}(\Lambda)$ *and $(L, h_L)|\Gamma_j$ is big for every j. If A is an integral nef divisor on X, then the sections in*

$$H^0(S, \mathcal{O}_S(K_S + A_S + \Gamma_S + L_S) \otimes \mathrm{Adj}_{\Gamma_S}(S, h_L|S)))$$

are in the image of the restriction:

$$H^0(X, \mathcal{O}_X(K_X + S + A + \Gamma + L)) \to H^0(S, \mathcal{O}_S(K_S + A_S + \Gamma_S + L_S)).$$

□

First we shall assume that Q_s is big. We shall replace D_s by

$$(D_s - \mu_{s,*}E_s)_{\geqq 0} \tag{5.6}$$

where for a Weil divisor $F = \sum a_i F_i$, $F_{\geqq 0} := \sum \max(a_i, 0) F_i$, $\mu_s : Y_s \to X_s$ be the morphism as in (5.4) and E_s is the effective $\mathbb{Q}$-divisor in (5.4). And we shall change the coefficients of D so that $D|X_s = D_s$ holds.

Let A be an ample $\mathbb{Q}$-divisor on X. Let k be a sufficiently divisible positive integer such that kD and kA are integral divisors. Now we proceed as [E-P]. First we shall decompose kD as

$$kD = \Delta_1 + \cdots + \Delta_{k-1} \tag{5.7}$$

such that each Δ_i is a reduced simple normal crossing divisor. This is possible, since (X, D) is KLT and $\mathrm{Supp}\, D$ is a simple normal crossing divisor. We set

$$M := kK_X + kA + \Delta_1 + \cdots + \Delta_{k-1}. \tag{5.8}$$

Let H be a sufficiently ample line bundle on X which such that for any pseudoeffective singular hermitian line bunlde (L, h_L) on X_s, $\mathcal{O}_{X_s}(H + L + (\ell+1)K_X + kA + \Delta_1 + \cdots + \Delta_\ell) \otimes \mathcal{I}(h_L)$ is globally generated for every $0 \leqq \ell \leqq k-1$. For $0 \leqq \ell \leqq k-1$, we extend every element of

$$H^0(X_s, \mathcal{O}_{X_s}(H + mM + (\ell+1)(K_X + X_s) + kA + \Delta_1 + \cdots + \Delta_\ell) \otimes \mathcal{I}(h_Q^{km})) \tag{5.9}$$

to an element of

$$H^0(X, \mathcal{O}_X(H + mM + (\ell+1)K_X + kA + \Delta_1 + \cdots + \Delta_\ell)) \tag{5.10}$$

by induction on m and ℓ. The assertion is trivial when $m = \ell = 0$. Suppose that we have already costructed the extension for some m and $\ell - 1$. Then we take a basis of $\{\sigma_{\ell,1}^{(m)}, \cdots, \sigma_{\ell,N(m,\ell)}^{(m)}\}$ of

$$H^0(X_s, \mathcal{O}_{X_s}(H + mM + \ell(K_X + X_s) + kA + \Delta_1 + \cdots + \Delta_\ell) \otimes \mathcal{I}(h_Q^{km})) \tag{5.11}$$

and extend the basis to a set of sections $\{\tilde{\sigma}_{\ell,1}^{(m)}, \cdots, \tilde{\sigma}_{\ell,N(m,\ell)}^{(m)}\}$ in

$$H^0(X, \mathcal{O}_X(H + mM + \ell K_X + kA + \Delta_1 + \cdots + \Delta_\ell)) \tag{5.12}$$

by Proposition 5.8. We define the singular hermitian metric $h_{m,\ell}$ on $mM + H + \ell K_X + \Delta_1 + \cdots + \Delta_\ell$ by

$$h_{m,\ell} := \frac{1}{\sum_{j=1}^{N(m,\ell)} |\tilde{\sigma}_{\ell,j}^{(m)}|^2} \tag{5.13}$$

which is apparently of algebraic singularities and with semipositive curvature. As in the proof of [E-P, Proposition 5.4], by the induction on ℓ, if we take H to be sufficiently ample, by the induction on ℓ, we see that

$$\mathcal{I}(h_Q^m) \subseteq \mathrm{Adj}_{\Delta_\ell}(X_s; h_{m,\ell}|X_s) \tag{5.14}$$

holds for every $0 \leqq \ell \leqq k-1$. We note that $\mathrm{Adj}_{\Delta_\ell}(X_s; h_{m,\ell}|X_s)$ is well defined, since by the choice of H, $\mathcal{O}_{X_s}((H + mM + (\ell+1)K_X + kA + \Delta_1 + \cdots + \Delta_\ell)|X_s) \otimes \mathcal{I}(h_Q^{km})$ is globally generated and the adjustment (5.6) implies $\mathrm{Bs}|((H+mM+(\ell+1)K_X+kA+\Delta_1+\cdots+\Delta_\ell)|X_s)\otimes\mathcal{I}(h_Q^{km})|$ does not contain any irreducible component of Γ. Moreover $|((H+mM+(\ell+1)K_X+kA+\Delta_1+\cdots+\Delta_\ell)|X_s)\otimes\mathcal{I}(h_Q^{km})|$ defines a birational map from each component of Γ into a projective space by the effect of the ample line bundles H and A[6]. Hence we may apply Proposition 5.8. This completes the inductive construction of the metrics $\{h_{m,\ell}\}_{\ell=0}^{k-1}$. Then we set $h_{m+1,0} = h_{m,k-1}$ and continue the induction. This completes the construction of the metrics $\{h_{m,\ell}\}_{m,\ell\geqq 0}$.

Let h_A be a C^∞-hermitian metric on A with strictly positive curvature. Let $\hat{h}_{can,D}(A, h_A)$ (resp. $\hat{h}_{can,D,s}((A, h_A)|X_s)$) be the supercanonical AZD on $K_{X/S} + A + D$ (resp. $K_{X_s} + A_s + D_s$) with respect to h_A similar to Theorem 3.9 in Section 3.6.

Hence by the similar argument as in the proof of Lemma 4.8 in Section 4.4, using Hölder's inequality, the existence of the sequece of metrics $\{h_{m,\ell}\}_{m,\ell\geqq 0}$ implies that

$$\hat{h}_{can,D}(A, h_A)|X_s = O(\hat{h}_{can,D,s}(A, h_A)|X_s) \tag{5.15}$$

holds. Replacing (A, h_A) by $(\epsilon A, h_A^\epsilon)(\epsilon \in \mathbb{Q}^+)$ and letting ϵ tend to 0, by the argument in Section 4.5, we completes the proof of Lemma 5.4. □

The rest of the proof is similar as the one of Theorme 1.12. This completes the proof of Theorem 4.3. □

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